Introduction to the Jacobson radical, with a view towards representation theory

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This document gathers some elementary results about the Jacobson radical. Its main purpose is to show that a number of different definitions are equivalent and to demonstrate that the Jacobson radical is a double-sided ideal. After that, we will identify it in certain algebras arising from representation theory. All results presented here are well-known in the literature.

1 Basic Definitions

Throughout this document, a ring R will always be assumed to have a multiplicative identity element. That is, there exists an element $1 \in R$ such that $1x = x = x1$ for all $x \in R$. We also assume that $0 \neq 1$, so that R has at least two elements. We do not assume R to be commutative, however, which means that for $a, b \in R$ we may have $ab \neq ba$.

Our most important examples are *real algebras*. For our purpose, a real algebra is a ring A containing a specified copy of the real numbers $\mathbb{R}$ as a subring in its center. That is to say, for all $r \in \mathbb{R}$ we have an element $\phi_r \in A$ so that:

- $\phi_r = \phi_s$ for $r, s \in \mathbb{R}$ if and only if $r = s$;
- $\phi_1 = 1$ (the multiplicative identity element of A);
- $\phi_r + \phi_s = \phi_{r+s}$ and $\phi_r \phi_s = \phi_{rs}$ for all $r, s \in \mathbb{R}$;
- $\phi_r x = x \phi_r$ for all $r \in \mathbb{R}$ and $x \in A$.

One verifies that this makes A into a real vector space, where multiplication by a number $r \in \mathbb{R}$ is defined by $x \mapsto \phi_r x = x \phi_r$ for all $x \in A$. We will henceforth often write $rx := \phi_r x$, after which we may retrieve the element ϕ_r as $\phi_r = r1$ for 1 the multiplicative identity element of A.

Since the ring R is not assumed to be commutative, we need to distinguish between *left ideals*, *right ideals* and *two-sided ideals*. A left ideal is a non-empty subset $I \subseteq R$ such that for all $u, v \in I$ and $x \in R$ we have $-u, u + v, xu \in I$. Analogously, a right ideal is a non-empty subset $I \subseteq R$ such that for all $u, v \in I$

and $x \in R$ we have $-u, u+v, ux \in I$. A two-sided ideal is a subset of R that is both a left- and a right ideal. Note that in a real algebra A, any right-, left- or two-sided ideal I is also a linear subspace of A. This is because for any $r \in \mathbb{R}$ and $x \in I$ we have $rx = \phi_r x = x\phi_r \in I$.

Of special importance to us are the so-called maximal ideals.

Definition 1.1. *A* maximal left ideal *is a proper left ideal $I \subseteq R$ (proper means that $I \neq R$), such that if we have a left ideal $J \subseteq R$ for which $I \subseteq J \subseteq R$, then necessarily $J = I$ or $J = R$.*

Likewise, a maximal right ideal *is a right ideal $I \subsetneq R$ such that no right ideal J exists satisfying $I \subsetneq J \subsetneq R$.*

A maximal two-sided ideal *is a two-sided ideal $I \subsetneq R$ such that no other two-sided ideal sits between I and R.*

Maximal left-, right- and two-sided ideals always exist in any ring R (provided we have a multiplicative identity element), due to Zorn's lemma. More precisely, given any (left-, right- or two-sided) proper ideal $I \subsetneq R$, there exists a maximal (resp. left-, right- or two-sided) ideal J such that $I \subseteq J$. In the case of finite rings or finite-dimensional algebras, this can be shown with a much simpler argument. Namely, let $\mathcal{S}_I$ be the collection of all proper ideals containing I, and choose any $J \in \mathcal{S}_I$ with highest cardinality or dimension. Note that $\mathcal{S}_I$ is non-empty as $I \in \mathcal{S}_I$. Then J is necessarily maximal and $I \subseteq J$ by assumption. Thus, to show that maximal ideals exist, it remains to show that R contains at least one proper, two-sided ideal I. To this end, we may simply choose $I = \{0\}$.

Maximal ideals are closely related to so-called simple modules, as is shown in Lemma 1.3 below. Recall that a left module M over a ring R is an abelian group (with the group operation denoted by $+$ and with identity $0 \in M$), together with an operation $\cdot\colon R \times M \to M$ such that

- $1 \cdot m = m$ for all $m \in M$;
- $s \cdot (r \cdot m) = (sr) \cdot m$ for all $s, r \in R$ and $m \in M$;
- $(s+t) \cdot m = s \cdot m + t \cdot m$ for all $s, r \in R$ and $m \in M$;
- $s \cdot (m+n) = s \cdot m + s \cdot n$ for all $s \in R$ and $m, n \in M$.

Similarly, a right module M over a ring R is an abelian group with an operation $\cdot\colon M \times R \to M$ such that

- $m \cdot 1 = m$ for all $m \in M$;
- $(m \cdot r) \cdot s = m \cdot (rs)$ for all $s, r \in R$ and $m \in M$;
- $m \cdot (s+t) = m \cdot s + m \cdot t$ for all $s, r \in R$ and $m \in M$;
- $(m+n) \cdot s = m \cdot s + n \cdot s$ for all $s \in R$ and $m, n \in M$.

As with rings, we will henceforth denote the operation '$\cdot$' simply by concatenation. In case the ring R is a real algebra, any left- or right module is also a real vector space. A *submodule* of a left module M is a non-empty subset $N \subseteq M$ such that for all $m, n \in N$ and $r \in R$ we have $-m, m+n, rn \in N$, and we similarly define a submodule of a right module. As there is in general no notion of right-multiplication in a left module or vice versa, there is no ambiguity in writing 'submodule' for both notions in a left- or right module. In case such ambiguity does exist, we may sometimes write 'left submodule' or 'right submodule' instead.

Definition 1.2. *A left- or right module M over the ring R is called* simple *(or sometimes* irreducible*) if $M \neq \{0\}$ and the only submodules of M are $\{0\}$ and M itself.*

Given a left module M over a ring R and a submodule $N \subseteq M$, we may form the quotient M/N, which is again a left module over R. The same construction of course works for right modules. Note that a ring R is naturally both a left- and right module over itself. Its left (resp. right) submodules are then precisely its left (resp. right) ideals. Thus, a left ideal I over R gives rise to the left module R/I, and similarly for right ideals. The following result uses this construction to relate maximal ideals to simple modules.

Lemma 1.3. *Let $I \subseteq R$ be a left (resp. right) ideal. The left (resp. right) module R/I is simple, if and only if I is maximal.*

Proof. We restrict the proof to left ideals and modules. The case for the right objects goes precisely the same. Suppose first that R/I is simple, and let J be a left ideal such that $I \subseteq J \subseteq R$. We consider the submodule $J/I := \{[x] \mid x \in J\} \subseteq R/I$, where $[x] \in R/I$ denotes the class of $x \in R$. Because R/I is assumed simple, there are two options. Either $J/I = \{0\}$ or $J/I = R/I$. In the first case we have $J \subseteq I$ and so $J = I$. In the second case we see that for all $r \in R$ there is an $x \in J$ such that $[r] = [x] \in R/I$. Thus we have $r - x \in I \subseteq J$, but because also $x \in J$ we find $r = (r-x) + x \in J$. This shows that $J = R$. It remains to show that I does not equal R. However, if this were the case then we would find $R/I = \{0\}$, contradicting the definition of a simple module. This shows that I is indeed maximal.

Now suppose R/I is not simple. Then either $R/I = \{0\}$, in which case $I = R$ is not maximal, or there is a submodule $\{0\} \subsetneq N \subsetneq R/I$. In the latter case, consider the set

$$J_N := \{r \in R \mid [r] \in N\}\,. \tag{1}$$

Given $r, s \in R$ with $[r] \in N$, it follows that $[sr] = s[r] \in N$. Using this, one verifies that J_N is a left ideal of R. If $J_N = R$ then $1 \in J_N$ and so $[1] \in N$. This in turn implies that $r[1] = [r] \in N$ for all $r \in R$, and so $N = R/I$. From this contradiction we learn that instead $J_N \subsetneq R$. Next, we note that for all $r \in I$ we have $[r] = [0] \in N$ and thus $I \subseteq J_N$. Finally, to show that $I \neq J_N$ pick any non-zero element $n \in N$. It holds that $n = [r]$ for some $r \in R$ and so necessarily $r \in J_n$. But since $[r] = n \neq 0$, we see that $r \notin I$. This shows that $I \subsetneq J_N$. Summarizing, we find $I \subsetneq J_N \subsetneq R$, so that I is not maximal. ∎

Another useful notion is that of a homomorphism between left R-modules M and M'. This is a function $f : M \to M'$ satisfying

- $f(m+n) = f(m) + f(n)$ for all $m, n \in M$;
- $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$.

Note that $f(0) = f(0+0) = f(0) + f(0)$, from which

$$0 = f(0) - f(0) = f(0) + f(0) - f(0) = f(0)\,.$$

In turn, we find $f(m)+f(-m) = f(m-m) = f(0) = 0$ and so $f(-m) = -f(m)$. Or note alternatively that $(1-1)m = m + (-1)m = 0$, so that $(-1)m = -m$. Thus we retrieve $f(-m) = f((-1)m) = (-1)f(m) = -f(m)$.

If the homomorphism $f\colon M \to M'$ is bijective then of course an inverse function $f^{-1}\colon M' \to M$ exists. For this function we have

$$f^{-1}(f(m) + f(n)) = f^{-1}(f(m+n)) = m + n = f^{-1}(f(m)) + f^{-1}(f(n))$$

and

$$f^{-1}(rf(m)) = f^{-1}(f(rm)) = rm = rf^{-1}(f(m))\,,$$

for all $m, n \in M$ and $r \in R'$. As f is surjective, so that $f(m)$ and $f(n)$ vary over M' as m and n vary over M, we conclude that f^{-1} is a homomorphism as well. If a bijective homomorphism $f\colon M \to M'$ exists (and so equivalently a bijective homomorphism $g\colon M' \to M$), we say that M and M' are *isomorphic*. Such a bijective f is called an *isomorphism*.

In the same way we have homomorphisms between right R-modules, with the corresponding analogues properties.

Arguably the most important result on homomorphisms is the *first isomorphism theorem*.

Lemma 1.4 (First isomorphism theorem)**.** *Let $f\colon M \to M'$ be a homomorphism between (left- or right-) R-modules M and M'. Then the kernel $\ker(f) \subseteq M$ and image $\mathrm{Im}(f) \subseteq M'$ are submodules, and there exists an isomorphism*

$$[f]\colon M/\ker(f) \cong \mathrm{Im}(f)\,,$$

given explicitly by $[f]([x]) = f(x)$, for $[x] \in M/\ker(f)$ with $x \in M$.

Proof. It is a standard exercise to show that $\ker(f)$ and $\mathrm{Im}(f)$ are submodules of M and M', respectively. To show that $[f]\colon M/\ker(f) \to \mathrm{Im}(f)$ is well-defined, note that indeed $f(x) \in \mathrm{Im}(f)$. If we have $[x] = [y] \in M/\ker(f)$ for some $x, y \in M$, then $x - y \in \ker(f)$ and so $f(x) - f(y) = f(x-y) = 0$. Thus the definition $[f]([x]) := f(x)$ is unambiguous. One then easily verifies that $[f]$ is a homomorphism. Clearly $\mathrm{Im}([f]) = \mathrm{Im}(f)$, and if $[f]([x]) = f(x) = 0$ then $x \in \ker(f)$ and so $[x] = 0$. This shows that $[f]$ is indeed an isomorphism. $\square$

2 The Jacobson Radical and Local Rings

We now present the main result of this document, which shows that a number of definitions are equivalent. These describe the so-called Jacobson radical.

Theorem 2.1. *For $x \in R$ the following are equivalent.*

1L. Every maximal left ideal contains x.

2L. The element x is in the annihilator of every simple left module of R. That is, for every simple left module M of R and every $m \in M$, we have $xm = 0$.

3L. For every $r \in R$, the element $1 - rx$ has a left inverse. In other words, there is a $u \in R$ such that $u(1 - rx) = 1$.

4. For all $r, s \in R$, the element $1 - rxs$ has a double-sided inverse. In other words, there is a $u \in R$ such that $u(1 - rxs) = (1 - rxs)u = 1$.

1R. Every maximal right ideal contains x.

2R. The element x is in the annihilator of every simple right module of R. That is, for every simple right module M of R and every $m \in M$, we have $mx = 0$.

3R. For every $r \in R$, the element $1 - xr$ has a right inverse.

Proof. [$1L \implies 2L$]: Suppose that every maximal left ideal contains x, and fix a simple left module M. Let $m \in M$ be given. If $m = 0$ then clearly $xm = 0$, and so we may assume that $m \neq 0$. Consider the function

$$\begin{aligned} f_m &: R \to M \\ f_m(r) &= rm\,. \end{aligned} \tag{2}$$

One verifies that if we view R as a left R-module, then f_m becomes a homomorphism. We denote its image by $Rm \subseteq M$. Since $0 \neq m = 1m \in Rm$, we see that $Rm \neq \{0\}$. Thus, since M is simple, it must hold that $Rm = M$ and so f_m is surjective. Denote by $I \subseteq R$ the kernel of f_m. Then I is a submodule of R, with the latter seen as a left R-module, and so I is a left ideal of R. We conclude by the first isomorphism theorem (Lemma 1.4) that we have an isomorphism of left R-modules

$$\begin{aligned} [f_m] &: R/I \to M \\ [f_m]([r]) &= rm\,, \end{aligned} \tag{3}$$

where $[r] \in R/I$ denotes the class of $r \in R$. Since M is a simple module, so is R/I. We therefore conclude from Lemma 1.3 that I is a maximal left-ideal. In particular, we find $x \in I$. This implies that $[0] = [x] \in R/I$, and so

$$xm = [f_m]([x]) = [f_m]([0]) = 0\,. \tag{4}$$

$[2L \implies 1L]$: Now suppose x is in the annihilator of every simple left module of R. Let I be a given maximal left ideal of R. From Lemma 1.3 we know that R/I is a simple left module. Thus, $xm = 0$ for all $m \in R/I$. In particular, take $m = [1]$. Then $0 = x[1] = [x]$, from which it follows that $x \in I$.

$[1L \implies 4]$ By the previous steps, we know that 1L and 2L are equivalent. In particular, if x is contained in every maximal left ideal of R, then for every simple left module M and every $m \in M$, we have $xm = 0$. Thus, given any $r, s \in R$ and m in some simple left module M, we see that $(rxs)m = r(x(sm)) = r0 = 0$, as $sm \in M$. We therefore find that rxs is likewise in the annihilator of any simple left module, and so is contained in every maximal left ideal of R. Thus, we only need to show that $1 - x$ has a two-sided inverse, for every x that is contained in the intersection of all maximal left ideals of R.

We first show that a left-inverse exists, so that $u(1-x) = 1$ for some $u \in R$. Suppose otherwise, then the left ideal $R(1-x) := \{r(1-x) \mid r \in R\}$ does not contain 1, and so is proper. It follows that a maximal left ideal I exists such that $R(1-x) \subseteq I$. In particular, we have $1-x \in I$. Since by assumption we also have $x \in I$, we find $1 = (1-x) + x \in I$. This contradicts that I is maximum, and so indeed $u(1-x) = 1$ for some $u \in R$.

Rewriting $u(1-x) = 1$, we find $u - ux = 1$ and so $u = 1 + ux = 1 - (-ux)$. Since $-ux$ is likewise contained in the intersection of all maximal left ideals, we may use the same argument as before to conclude that $v(1 - (-ux)) = vu = 1$ for some $v \in R$. Combined with the fact that $u(1-x) = 1$, we find

$$v = v1 = v(u(1-x)) = (vu)(1-x) = 1 - x\,. \tag{5}$$

Thus $(1-x)u = u(1-x) = 1$, which proves that $1 - x$ indeed has a two-sided inverse.

$[4 \implies 3L]$ This follows immediately by setting $s = 1$.

$[3L \implies 1L]$ Suppose I is a maximal left ideal not containing x. Then the left ideal $Rx + I := \{rx + y \mid r \in R \text{ and } y \in I\}$ strictly contains I (as $x \in Rx + I$ but $x \notin I$). Since I is maximal, we find $Rx + I = R$. In particular, some $r \in R$ and $y \in I$ exist such that $rx + y = 1$. Rewriting this expression gives $1 - rx = y \in I$. But by assumption, $1 - rx$ has a left-inverse, so that $1 \in R(1 - rx) \in I$. This contradicts the assumption that I is a maximal, and therefore proper ideal, and we conclude that x has to be contained in every maximal left ideal of R.

The proofs for $1R \implies 2R, 2R \implies 1R, 1R \implies 4, 4 \implies 3R$ and $3R \implies 1R$ are completely analogous, so that the result follows. $\square$

Definition 2.2. *Given a ring R, the set of all elements $x \in R$ satisfying one of the equivalent statements of Theorem 2.1 is called the* Jacobson radical *of R.*

Note that by points 1R and 1L, the Jacobson radical is a two-sided ideal. Next, we explore what it means for a ring to have only one right (or left) maximal ideal.

Theorem 2.3. *A ring has only one maximal left ideal, if and only if it has only one maximal right ideal.*

Proof. We will only show that if a ring R has a single maximal left ideal, it has a single maximal right ideal. The reverse direction is fully analogues. Thus, suppose R has a single maximal left ideal J. By Theorem 2.1, J is the Jacobson radical of R and is therefore two-sided. This means R/J has a well-defined ring structure, with multiplication given by $[x][y] = [xy]$ for $[x], [y] \in R/J$ the classes of $x, y \in R$, respectively. Note also that $[0] \neq [1]$, as $1 \notin J$.

We claim that every non-zero element of R/J has a two-sided inverse. To this end, let $[x] \in R/J$ be given such that $[x] \neq [0]$. It follows that $x \notin J$, so that the left-ideal $J + Rx$ strictly contains J. By maximality of this latter ideal, we find that $J + Rx = R$. Thus, there exist $y \in R$ and $m \in J$ such that $1 = m + yx$. This in turn implies that $[y][x] = [yx] = [1 - m] = [1] \in R/J$, as $m \in J$. To show that likewise $[x][y] = [1]$, note that $y \notin J$, as otherwise we would arrive at the contradiction $[1] = [y][x] = [0][x] = [0]$. Thus in precisely the same way as for $[x]$, we see that there is an element $[z] \in R/J$ such that $[z][y] = [1]$. But then

$$[z] = [z][1] = [z]([y][x]) = ([z][y])[x] = [1][x] = [x]\,.$$

We see that $[y][x] = [1] = [z][y] = [x][y]$, so that $[x]$ indeed has the two-sided inverse $[y]$.

Now let $K \subseteq R$ be any maximal right ideal. By definition of the Jacobson radical — more precisely point 1R in Theorem 2.1 — we see that $J \subseteq K$. If $J \neq K$, then we may pick an element $x \in K \setminus J$. It follows that $[x]$ is a non-zero element of R/J, so that a two-sided inverse $[y] \in R/J$ exists. In particular, we have $[x][y] = [1]$. This means that $xy - 1 \in J \subseteq K$. However, since K is a right ideal containing x, we have $xy \in K$ and so $xy - (xy - 1) = 1 \in K$. This contradicts the fact that K is maximal, and we conclude that instead $J = K$. Thus, the only maximal right ideal of R is J, which proves the claim. □

Theorem 2.3 motivates the following definition.

Definition 2.4. *A ring with only one maximal left ideal (equivalently only one maximal right ideal) is called a* local ring.

Note that for any local ring, the unique maximal left ideal is necessarily equal to its Jacobson radical, by Theorem 2.1. Similarly, its unique maximal right ideal is given by its Jacobson radical and thus all three ideals are equal and are two-sided.

Lemma 2.5. *For R a local ring with Jacobson radical J, we may write*

$$\begin{aligned} J &= \{x \in R \mid x \textit{ has no two-sided inverse } \} \qquad (6)\\ &= \{x \in R \mid x \textit{ has no left inverse } \}\\ &= \{x \in R \mid x \textit{ has no right inverse } \}\,. \end{aligned}$$

Proof. If $x \in R$ has a two-sided inverse, then it clearly also has both a left and a right inverse. Now suppose x has a left inverse. That is, there exists a $y \in R$ such that $yx = 1$. If we have $x \in J$ then likewise $1 = yx \in J$, contradicting the fact that J is a maximal ideal. Note that these conclusions still hold if we drop the assumption that R is local. We only need that $1 \notin J$, which holds for any ring as the Jacobson radical J is contained in at least one maximal left ideal. Thus, if x has a left inverse then $x \notin J$, and of course the same conclusion holds if x has a right inverse.

Now suppose we have $x \notin J$ and consider the left ideal $Rx \subseteq R$. We either have that $Rx = R$, or that Rx is contained in a maximal left ideal, which is then necessarily J (here we use that R is local). This latter option is excluded as $x \notin J$. Thus $Rx = R$ and so $ux = 1$ for some $u \in R$. Analogously, we find $xv = 1$ for some $v \in R$. Finally we see that $u = u1 = u(xv) = (ux)v = v$, so that x has the two-sided inverse $u = v$. This completes the proof. □

We may now give an alternative characterization of a local ring.

Proposition 2.6. *Given a ring R, the following are equivalent:*

1. *The ring R is local;*
2. *Whenever $x, y \in R$ both do not have a left inverse, their sum $x + y$ also has no left inverse;*
3. *Whenever $x, y \in R$ both do not have a right inverse, their sum $x + y$ also has no right inverse.*

Proof. Let us denote by S_L and S_R the set of elements in R that have no left inverse and those that have no right inverse, respectively. If we assume point 1 to hold, then Lemma 2.5 tells us that S_R and S_L are both equal to the Jacobson radical. In particular both 2 and 3 then hold true.

Now assume 2 to hold. We will first show that S_L is a left ideal. To this end, note that clearly $0 \in S_L$. Suppose we are given $x \in S_L$ and $r \in R$. If rx has a left inverse u then $1 = u(rx) = (ur)x$, contradicting that $x \in S_L$. Thus we find $rx \in S_L$. By assumption, S_L is closed under addition, so that it is indeed a left ideal. Next, note that S_L is proper, as $1 \notin S_L$. We claim that S_L is the unique maximal left ideal of R. To see why, let $I \subsetneq R$ be any proper left ideal of R. If I contains any element with a left inverse, then $R = RI = I$, contradicting our assumption that I is proper. Thus, $I \subseteq S_L$. This proves that S_L is maximal, and moreover tells us that S_L is the unique maximal left ideal of R, as S_L has to contain every maximal ideal. Thus R has a unique maximal left ideal, and so is indeed local. Showing that 3 implies 1 is of course analogous. □

Recall that an element $x \in R$ is called *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$. In representation theory, one often encounters rings where every element either has a left inverse, or is nilpotent. In that case an element cannot be both. For if $x^n = 0$ and $ux = 1$, then $1 = u^n x^n = u^n 0 = 0$ which contradicts our assumptions on R. Such rings are necessarily local by the following proposition.

Proposition 2.7. *If a ring R has the property that every element either has a left inverse or is nilpotent then it is local. The same holds true if every element either has a right inverse or is nilpotent. In both cases, the Jacobson radical* *consists of precisely all the nilpotent elements, and the other elements have two-sided inverses.*

Proof. We only show the case where every element either has a left inverse or is nilpotent, the case for right inverses is similar. As an element cannot both be nilpotent and have a left inverse, Proposition 2.6 tells us that R is local if the sum of any two nilpotent elements is again nilpotent. Therefore, let $x, y \in R$ be nilpotent and let $r \in R$ be any element. We first claim that rx is again nilpotent. Suppose otherwise, then rx has a left-inverse, say $s \in R$. This gives us $1 = s(rx) = (sr)x$, contradicting that x has no left inverse. Now suppose that $x + y$ has a left inverse, say $u \in R$. Then $ux + uy = u(x + y) = 1$, and so $y' = 1 - x'$, where $x' := ux$ and $y' := uy$ are both nilpotent by our previous claim. However, if $x'^n = 0$ then

$$\begin{aligned} &(1 + x' + x'^2 + \cdots + x'^{n-1})y' \\ =&(1 + x' + x'^2 + \cdots + x'^{n-1})(1 - x') \\ =&1 + x' + x'^2 + \cdots + x'^{n-1} - (x' + x'^2 + \cdots + x'^n) \\ =&1 - x'^n = 1\,. \end{aligned} \tag{7}$$

This contradicts the fact that y' is nilpotent, and we conclude that $x + y$ has to be nilpotent instead. Thus we see that R is indeed a local ring. By Lemma 2.5, the Jacobson radical consists of all the nilpotent elements, whereas all other elements have two-sided inverses. □

In the following section we need:

Lemma 2.8. *Let R be a local ring with Jacobson radical J. Every simple left R-module is isomorphic to R/J (as left modules) and every simple right R-module is isomorphic to R/J (as right modules).*

Proof. Let M be a simple left R-module. As in the proof of Theorem 2.1, we find a left ideal $I \subseteq R$ such that M is isomorphic to R/I as left modules. By Lemma 1.3, the left ideal I is maximal and so necessarily equal to J. Thus $M \cong R/J$. The proof for right modules is similar. □

We finish this section with a result that is useful though intuitively clear.

Lemma 2.9. *Let R be a ring with Jacobson radical J. The Jacobson radical of the quotient ring $S = R/J$ equals $\{0\}$.*

Proof. Let $x \in S$ be contained in the Jacobson radical of S. We may write x as the class of some element $r \in R$, so $x = [r]$. By point 3L of Theorem 2.1, for any $t \in R$ the element $[1] - [t][r] \in S$ has a left inverse $[u]$. Thus we have $[1] = [u]([1] - [t][r]) = [u] - [utr]$, which implies that $1 - u + utr \in J$. Again by Theorem 2.1, this means that $1 - (1 - u + utr) = u - utr$ has a left inverse, say that $v(u - utr) = 1$. But then we find $(vu)(1 - tr) = 1$, which shows that $1 - tr$ has a left inverse. This result holds for any choice of $t \in R$, and so we conclude that $r \in J$. In other words, we find $[r] = [0]$. As $x = [r]$ was chosen arbitrarily from the Jacobson radical of S, the result follows. □

3 Algebras and Matrix-Rings

We now have a closer look at algebras and rings of matrices. The first notion we need is:

Definition 3.1. *Let M be a left module over the ring R. A* composition series *of M is a finite set of submodules $M_0, \dots, M_k \subseteq M$ such that*

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M$$

with the property that no submodules can be inserted. In other words, if $N \subseteq M$ is a submodule such that $M_{i-1} \subseteq N \subseteq M_i$ for some $i \in \{1, \dots, k\}$, then $N = M_{i-1}$ or $N = M_i$.

The same notion of course exists for right modules.

Lemma 3.2. *Let A be a real algebra and M a finite dimensional left module over A. Then M has a composition series.*

Proof. We prove the statement by induction on the dimension of M. If $M = \{0\}$ then a composition series is given by $M_0 = M$ and if M is one-dimensional then a composition series is given by $\{0\} \subsetneq M$. So, suppose a composition series exists for any left module of dimension ℓ or less, for some $\ell > 0$. Let M have dimension $\ell + 1$ and let $N \subsetneq M$ be any proper submodule of M with maximal dimension among all proper submodules. Then no submodule can sit between N and M. Moreover, N has dimension ℓ or less, and thus has a composition series

$$\{0\} = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = N\,.$$

We then get a composition series for M, given by

$$\{0\} = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = N \subsetneq M\,,$$

which proves the lemma. □

The following observation gives another way of viewing composition series.

Lemma 3.3. *Let M be a left module over a ring R with proper submodule $N \subsetneq M$. The quotient module M/N is simple, if and only if there is no submodule K that sits strictly between N and M, $N \subsetneq K \subsetneq M$.*

Proof. Note that the quotient M/N is non-zero, as $N \neq M$. If we have a submodule K satisfying $N \subsetneq K \subsetneq M$, then M/N has the non-zero, proper submodule K/N. Thus, M/N is not simple.

Now suppose conversely that M/N is not simple. Say we have a non-zero, proper submodule $\{0\} \subsetneq P \subsetneq M/N$. Consider the set

$$Q = \{m \in M \mid [m] \in P\}\,,$$

where $[m] \in M/N$ denotes the class of m. One verifies that Q is a submodule of M. Moreover, as $[n] = [0] \in P$ for all $n \in N$, we see that $N \subseteq Q$. Moreover, if $p \in P$ is any non-zero element, then we may write $p = [m]$ for some $m \in M$. Necessarily $m \notin N$, as $p \neq 0$, and so we find $N \subsetneq Q$. Equality between Q and M would imply $P = M/N$, and so instead we have $Q \subsetneq M$. Thus we find the submodule Q sitting strictly in between N and M. □

Next, we obtain an important consequence of the existence of a composition series.

Proposition 3.4. *Let A be a real algebra that is also a local ring. Denote by J the Jacobson radical of A and suppose the quotient ring A/J is finite dimensional as a real vector space. Given any finite dimensional left module M over A, the real dimension of A/J divides that of M.*

Proof. Let us denote the dimension of A/J, seen as a real vector space, by ℓ. First of all, by Lemma 3.2 M has a composition series, say

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M\,.$$

Next, it follows from Lemma 3.3 that every quotient module M_i/M_{i-1}, for $i \in \{1, \dots, k\}$, is simple. Thus by Lemma 2.8, each of these is isomorphic to A/J. This tells us in particular that $M_1 \cong M_1/M_0 \cong A/J$ has real dimension ℓ. Finally, we prove by induction that $\dim(M_i) = i\ell$ for all $i \in \{1, \dots, k\}$. To this end, suppose that $\dim(M_{i-1}) = (i-1)\ell$. It follows that

$$\begin{aligned} \dim(M_i) &= \dim(M_{i-1}) + \dim(M_i/M_{i-1}) \\ &= \dim(M_{i-1}) + \dim(A/J) = (i-1)\ell + \ell = i\ell\,. \end{aligned} \tag{8}$$

We therefore find $\dim(M) = \dim(M_k) = k\ell$, which completes the proof. □

The analogous statement of Proposition 3.4 for right modules of course holds as well. The following examples appear frequently. In fact, it can be shown that, together with $A/J \cong \mathbb{R}$, these are the only possibilities.

Example 3.5. *Suppose the real algebra A is a local ring with Jacobson radical J. If $A/J \cong \mathbb{C}$ as real algebras, then every finite dimensional (left or right) module over A has even dimension. If instead $A/J \cong \mathbb{H}$ (the quaternions), then the dimension of every module over A, when finite, is divisible by* 4.

We next prepare for Nakayama's lemma, which will give us more properties of the Jacobson radical of a finite dimensional algebra.

Given a ring R, we may consider the set $\mathrm{Mat}(R; n)$ of $n \times n$ matrices $B = (b_{i,j})$ with coefficients in R. If we define addition to be entry-wise (i.e. $(B+C)_{i,j} = B_{i,j} + C_{i,j}$) and multiplication as is usual for matrices (so $(BC)_{i,j} = \sum_{k=1}^{n} B_{i,k}C_{k,j}$) then this makes $\mathrm{Mat}(R; n)$ into a ring. The multiplicative identity element of $\mathrm{Mat}(R; n)$ is given by the identity matrix Id, defined by $\mathrm{Id}_{i,j} = \delta_{i,j}$. Here $\delta_{i,j} = 1$ if and only if $i = j$, with $\delta_{i,j} = 0$ otherwise.

Given any subset I of R, we may define the set $\mathrm{Mat}(I; n) \subseteq \mathrm{Mat}(R; n)$ as consisting of those matrices that have all of their coefficients in I. It is not hard to show that $\mathrm{Mat}(I;n)$ becomes a left or right ideal of $\mathrm{Mat}(R;n)$ if I is a left or right ideal of R, respectively.

The following result tells us that certain matrices in $\mathrm{Mat}(R;n)$ always have two-sided inverses.

Lemma 3.6. *Let R be a ring with Jacobson radical J and let $N \in \mathrm{Mat}(J;n)$ be given. There exists a matrix $B \in \mathrm{Mat}(J;n)$ such that $(\mathrm{Id}+B)(\mathrm{Id}+N) = (\mathrm{Id}+N)(\mathrm{Id}+B) = \mathrm{Id}$.*

Proof. It suffices to show that $B \in \mathrm{Mat}(J;n)$ exists such that $(\mathrm{Id}+B)(\mathrm{Id}+N) = \mathrm{Id}$. To see why, note that the same argument applied to B then gives a $C \in \mathrm{Mat}(J;n)$ such that $(\mathrm{Id}+C)(\mathrm{Id}+B) = \mathrm{Id}$. But then

$$(\mathrm{Id}+N) = (\,(\mathrm{Id}+C)(\mathrm{Id}+B)\,)(\mathrm{Id}+N) = (\mathrm{Id}+C)(\,(\mathrm{Id}+B)(\mathrm{Id}+N)\,) = (\mathrm{Id}+C)\,.$$

Thus $(\mathrm{Id}+N)(\mathrm{Id}+B) = (\mathrm{Id}+C)(\mathrm{Id}+B) = \mathrm{Id}$.

Let us write $N = (n_{i,j})$, so that $(\mathrm{Id}+N)_{i,j} = n_{i,j}+\delta_{i,j}$. The proof essentially uses Gaussian elimination to construct B, though we need to be careful with inverting elements. However, as $n_{i,i} \in J$ for all $i \in \{1,\ldots,n\}$, point 4 of Theorem 2.1 tells us that the diagonal element $1+n_{i,i}$ has a two-sided inverse.

Let us first define $B_1 \in \mathrm{Mat}(J;n)$ by $(B_1)_{i,j} = \delta_{1,j} n_{i,j}(1+n_{1,1})^{-1}$. Note that indeed $(B_1)_{i,j} \in J$ for all $i,j \in \{1,\ldots,n\}$, as $n_{i,j} \in J$. It follows that

$$\begin{aligned}(B_1(\mathrm{Id}+N))_{i,j} &= \sum_{k=1}^{n} (B_1)_{i,k}(\mathrm{Id}+N)_{k,j} \\ &= \sum_{k=1}^{n} \delta_{1,k} n_{i,k}(1+n_{1,1})^{-1}(\mathrm{Id}+N)_{k,j} \\ &= n_{i,1}(1+n_{1,1})^{-1}(\mathrm{Id}+N)_{1,j}\,.\end{aligned} \tag{9}$$

Substituting $j=1$ in Equation (9) above, we obtain

$$\begin{aligned}(B_1(\mathrm{Id}+N))_{i,1} &= n_{i,1}(1+n_{1,1})^{-1}(\mathrm{Id}+N)_{1,1} \\ &= n_{i,1}(1+n_{1,1})^{-1}(1+n_{1,1}) = n_{i,1} = N_{i,1}\,.\end{aligned} \tag{10}$$

Next, we note that

$$\begin{aligned}(\mathrm{Id}-B_1)(\mathrm{Id}+N) &= (\mathrm{Id}+N) - B_1(\mathrm{Id}+N) \\ &= \mathrm{Id}+(N - B_1(\mathrm{Id}+N)) = \mathrm{Id}+N'\ ,\end{aligned} \tag{11}$$

where $N' := N - B_1(\mathrm{Id}+N)$. As $N, B_1 \in \mathrm{Mat}(J;n)$, we likewise find $N' \in \mathrm{Mat}(J;n)$. Moreover, it follows from Equation (10) that

$$N'_{i,1} = N_{i,1} - (B_1(\mathrm{Id}+N))_{i,1} = N_{i,1} - N_{i,1} = 0\,, \tag{12}$$

for all $i \in \{1,\ldots,n\}$. Summarizing, we find that $(\mathrm{Id}-B_1)(\mathrm{Id}+N)$ can again be written as $\mathrm{Id}+N'$, where $N' \in \mathrm{Mat}(J;n)$ and with $N'_{i,1} = 0$ for all $i \in \{1,\ldots,n\}$.

Let us therefore now assume we have a matrix $M = (m_{i,j}) \in \mathrm{Mat}(J;n)$ together with an $\ell \geq 2$ such that $M_{i,j} = 0$ for all $i \in \{1,\dots,n\}$ and $j < \ell$. We will show that a matrix $B_\ell \in \mathrm{Mat}(J;n)$ exists such that $(\mathrm{Id} - B_\ell)(\mathrm{Id} + M) = \mathrm{Id} + M'$, where $M' \in \mathrm{Mat}(J;n)$ satisfies $M'_{i,j} = 0$ for all $i \in \{1,\dots,n\}$ and $j \leq \ell$. To this end, we define B_ℓ by $(B_\ell)_{i,j} = \delta_{\ell,j} m_{i,j}(1 + m_{\ell,\ell})^{-1} \in J$. As before, we obtain

$$
\begin{aligned}
(B_\ell(\mathrm{Id} + M))_{i,j} &= \sum_{k=1}^{n} (B_\ell)_{i,k}(\mathrm{Id} + M)_{k,j} \\
&= \sum_{k=1}^{n} \delta_{\ell,k} m_{i,k}(1 + m_{\ell,\ell})^{-1}(\mathrm{Id} + M)_{k,j} \\
&= m_{i,\ell}(1 + m_{\ell,\ell})^{-1}(\mathrm{Id} + M)_{\ell,j} \, .
\end{aligned} \tag{13}
$$

Setting $j = \ell$, we get $(B_\ell(\mathrm{Id} + M))_{i,\ell} = m_{i,\ell}(1 + m_{\ell,\ell})^{-1}(\mathrm{Id} + M)_{\ell,\ell} = m_{i,\ell}$. Moreover, for $j < \ell$ we have $M_{\ell,j} = 0$ and so $(\mathrm{Id} + M)_{\ell,j} = \mathrm{Id}_{\ell,j} = \delta_{\ell,j} = 0$. Thus for all $j < \ell$ and $i \in \{1,\dots,n\}$ we obtain $(B_\ell(\mathrm{Id} + M))_{i,j} = 0 = m_{i,j}$. In summary, we find $(B_\ell(\mathrm{Id} + M))_{i,j} = M_{i,j}$ for all $j \leq \ell$ and $i \in \{1,\dots,n\}$. Thus if we define $M' := M - B_\ell(\mathrm{Id} + M) \in \mathrm{Mat}(J;n)$ then $M'_{i,j} = M_{i,j} - M_{i,j} = 0$ for all $j \leq \ell$ and $i \in \{1,\dots,n\}$. Of course as before, $(\mathrm{Id} - B_\ell)(\mathrm{Id} + M) = \mathrm{Id} + M'$.

We see that we may inductively find matrices $B_1,\dots,B_n \in \mathrm{Mat}(J;n)$ such that

$$
(\mathrm{Id} - B_n)\dots(\mathrm{Id} - B_1)(\mathrm{Id} + N) = (\mathrm{Id} + K) \tag{14}
$$

for some $K \in \mathrm{Mat}(J;n)$ with all of its columns equal to zero. In other words $K = 0$, and so

$$
(\mathrm{Id} - B_n)\dots(\mathrm{Id} - B_1)(\mathrm{Id} + N) = \mathrm{Id} \, . \tag{15}
$$

Since we may write $(\mathrm{Id} - B_n)\dots(\mathrm{Id} - B_1) = \mathrm{Id} + B$ for some $B \in \mathrm{Mat}(J;n)$, the proof is complete. $\square$

As a corollary, we obtain:

Proposition 3.7. *Given a ring R with Jacobson radical J, the Jacobson radical of* $\mathrm{Mat}(R;n)$ *equals* $\mathrm{Mat}(J;n)$.

Proof. Let $X \in \mathrm{Mat}(J;n)$ and $B, C \in \mathrm{Mat}(R;n)$ be given. As $\mathrm{Mat}(J;n)$ is a two-sided ideal, we likewise have $-BXC \in \mathrm{Mat}(J;n)$. But then Lemma 3.6 tells us that $\mathrm{Id} - BXC = \mathrm{Id} + (-BXC)$ has a two-sided inverse in $\mathrm{Mat}(R;n)$. By point 4 of Theorem 2.1, we see that X is contained in the Jacobson radical of $\mathrm{Mat}(R;n)$.

Now suppose X is contained in the Jacobson radical of $\mathrm{Mat}(R;n)$. We need to show that $X \in \mathrm{Mat}(J;n)$, and so that $X_{i,j} \in J$ for all $i,j \in \{1,\dots,n\}$. To this end, we fix an index pair (k,ℓ) with $k,\ell \in \{1,\dots,n\}$. If we can show that for any $r \in R$, $1 - rX_{k,\ell}$ has a left-inverse in R, then point 3L of Theorem 2.1 indeed tells us that $X_{k,\ell} \in J$. Thus we now fix $r \in R$ as well, and define the matrices $B, C \in \mathrm{Mat}(R;n)$ by $B_{i,j} = r\delta_{1,i}\delta_{k,j}$ and $C_{i,j} = \delta_{\ell,i}\delta_{1,j}$ for all $i, j \in \{1, \dots, n\}$. It follows that

$$\begin{aligned}(BXC)_{i,j} &= \sum_{p=1}^{n}\sum_{q=1}^{n} B_{i,p}X_{p,q}C_{q,j} = \sum_{p=1}^{n}\sum_{q=1}^{n} r\delta_{1,i}\delta_{k,p}X_{p,q}\delta_{\ell,q}\delta_{1,j} \\ &= r\delta_{1,i}X_{k,\ell}\delta_{1,j} = rX_{k,\ell}\delta_{1,i}\delta_{1,j}\,.\end{aligned} \tag{16}$$

Since X is contained in the Jacobson radical of $\mathrm{Mat}(R;n)$, point 4 of Theorem 2.1 tells us that $\mathrm{Id} - BXC$ has a two-sided inverse $U \in \mathrm{Mat}(R;n)$. In particular for the $(1,1)$-entry, the identity $\mathrm{Id} = U(\mathrm{Id} - BXC)$ gives us

$$\begin{aligned}1 = (U(\mathrm{Id} - BXC))_{1,1} &= \sum_{p=1}^{n} U_{1,p}(\mathrm{Id} - BXC)_{p,1} \\ &= \sum_{p=1}^{n} U_{1,p}(\delta_{p,1} - rX_{k,\ell}\delta_{1,p}\delta_{1,1}) = \sum_{p=1}^{n} \delta_{p,1}U_{1,p}(1 - rX_{k,\ell}) \\ &= U_{1,1}(1 - rX_{k,\ell})\,.\end{aligned} \tag{17}$$

Thus $1 - rX_{k,\ell}$ indeed has a left-inverse, which shows that $X_{k,\ell} \in J$. We conclude that $X \in \mathrm{Mat}(J;n)$, which completes the proof. $\square$

The following famous lemma is extremely useful for showing that certain modules vanish. It involves so-called *finitely generated modules*. A left-module M over a ring R is called finitely generated if there exist finitely many elements $m_1, \dots, m_p \in M$ such that any element $m \in M$ can be written (not necessarily uniquely) as

$$m = r_1 m_1 + \dots + r_p m_p \tag{18}$$

for some $r_1, \dots, r_p \in R$. Note that if R is a real algebra with M finite-dimensional, then M is always finitely generated. To see why, choose for instance the generating set $\{m_1, \dots, m_p\} \subseteq M$ equal to some basis for M over the real numbers. In that case we may even choose all $r_1, \dots, r_p$ in Equation (18) from the copy of $\mathbb{R}$ in R.

In what follows, if M is a left-module over the ring R and I is a left-ideal of R, we denote by $IM \subseteq M$ the set consisting of all elements that may be written as finite sums of elements rm with $r \in I$ and $m \in M$. One readily shows that IM is a submodule of M.

Theorem 3.8 (Nakayama's lemma, non-commutative case)**.** *Let R be a ring with Jacobson radical J and M a finite generated left-module over R. If $JM = M$ then necessarily $M = 0$.*

Proof. We fix some generating set $\{m_1, \dots, m_p\}$ for M. As we allow the possibility that $m_i = 0$ for some i, we may assume that $p \geq 1$. From $M = JM$ we see that each m_i, for $i \in \{1, \dots, p\}$, may be written as

$$m_i = \sum_{k=1}^{N_i} r_{k,i}x_{k,i}\,, \tag{19}$$

where $N_i \in \mathbb{N}$, $r_{k,i} \in J$ and $x_{k,i} \in M$ for all $k \in \{1, \ldots, N_i\}$. Since $x_{k,i} \in M$ for all $i \in \{1, \ldots, p\}$ and $k \in \{1, \ldots, N_i\}$, we may further write

$$x_{k,i} = \sum_{j=1}^{p} a_{k,i,j} m_j \,, \tag{20}$$

for some $a_{k,i,j} \in R$. Putting equations (19) and (20) together, we obtain

$$m_i = \sum_{k=1}^{N_i} r_{k,i} \sum_{j=1}^{p} a_{k,i,j} m_j = \sum_{j=1}^{p} \left(\sum_{k=1}^{N_i} r_{k,i} a_{k,i,j} \right) m_j = \sum_{j=1}^{p} n_{i,j} m_j \,, \tag{21}$$

where

$$n_{i,j} := \sum_{k=1}^{N_i} r_{k,i} a_{k,i,j} \in J \,. \tag{22}$$

Thus, if we set $m := (m_1, \ldots, m_p)$ and define $N \in \operatorname{Mat}(J; p)$ by $N_{i,j} = n_{i,j}$, then we may conveniently summarize Equation (21) by $\operatorname{Id} m = Nm$, or

$$(\operatorname{Id} - N) m = 0 \,. \tag{23}$$

Now by Lemma 3.6, there exists a matrix $C \in \operatorname{Mat}(R; p)$ such that $C(\operatorname{Id} - N) = \operatorname{Id}$. Applying C to both sides of Equation (23) then gives us

$$m = \operatorname{Id} m = C(\operatorname{Id} - N) m = 0 \,. \tag{24}$$

Thus we find $m_i = 0$ for all $i \in \{1, \ldots, p\}$. As every element of M can be written as a combination of the m_i, we necessarily have $M = 0$, which completes the proof. □

Nakayama's lemma has strong consequences for the Jacobson radical of a finite dimensional real algebra. Just as we defined IM for a left ideal I and a module M, we may define the ideals I^2, I^3 and so forth. In general, given a two-sided ideal $I \subseteq R$, the two-sided ideal I^k consists of all finite sums of expressions $x_1 x_2 \ldots x_k$ with $x_1, \ldots, x_k \in I$. It is not hard to see that $I(I^k) = I^{k+1}$ for all $k > 0$.

Theorem 3.9. *Let A be a finite dimensional real algebra with Jacobson radical J. There exists a positive integer k such that $J^k = 0$.*

Remark 3.10. *Theorem 3.9 tells us that, for finite dimensional real algebras, every element of the Jacobson radical is nilpotent. More precisely, there exists a single $k \in \mathbb{N}$ (independent of x) such that $x^k = 0$ for all $x \in J$. In fact, given any $x_1, \ldots, x_k \in J$, we have $x_1 x_2 \ldots x_k = 0$.*

Proof of Theorem 3.9. Let us consider the descending chain of vector spaces

$$R \supseteq J \supseteq J^2 \supseteq \ldots \,.$$

Every time we have $J^p \supsetneq J^{p+1}$, the real dimension of J^{p+1} drops by at least 1 from that of J^p. As R is assumed finite dimensional, we see that for some $k \in \mathbb{N}$ we have

$$J^k = J^{k+1} = J(J^k)\,. \tag{25}$$

Note that J^k is a finitely generated module over A, as J^k is finite dimensional. Applying Nakayama's lemma (Theorem 3.8) to Equation (25) then indeed yields $J^k = 0$. $\square$