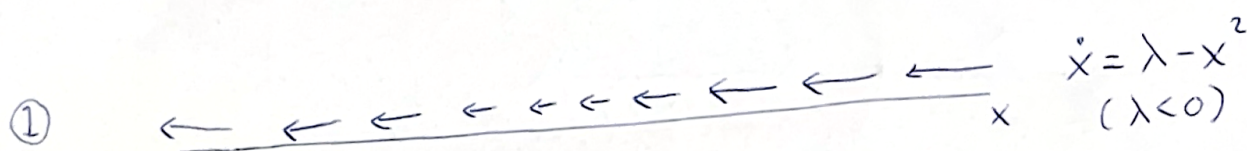
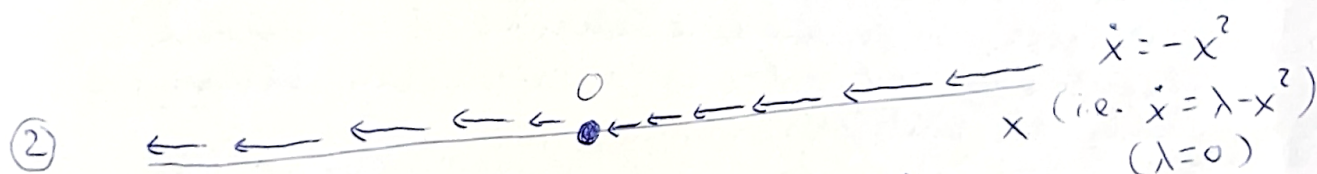


① Consider, for each fixed value of $\lambda \in \mathbb{R}$, the vectorfield f_λ on $\mathbb{R}$, defined by $f_\lambda(x) = \lambda - x^2$

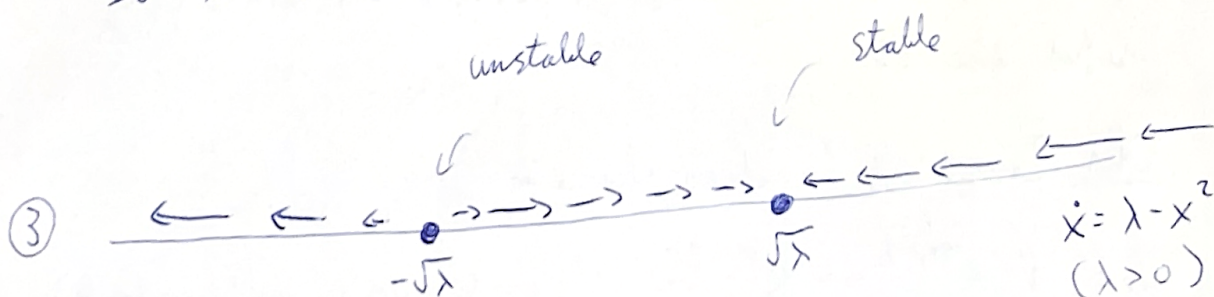
For $\lambda < 0$, a sketch of the phase portrait is given by



Though the exact formulas for the solutions vary in λ , the overall behavior, that is, the qualitative behavior, remains the same, as long as we keep λ (strictly) negative. Things look different for $\lambda = 0$:



For instance, where for $\lambda < 0$ every solution "shut off" to $-\infty$, we now have a steady state point at $x=0$. Moreover, solutions that start at a positive value of x limit to this steady-state point, and will of course not pass it. Things are again different for $\lambda > 0$. Now we have 2 steady-state points, found by solving $f_\lambda(x) = \lambda - x^2 = 0$. So $x^2 = \lambda \Rightarrow x = \pm\sqrt{\lambda}$. The picture is now



Note that $D_x f_\lambda(x) = -2x$, and so

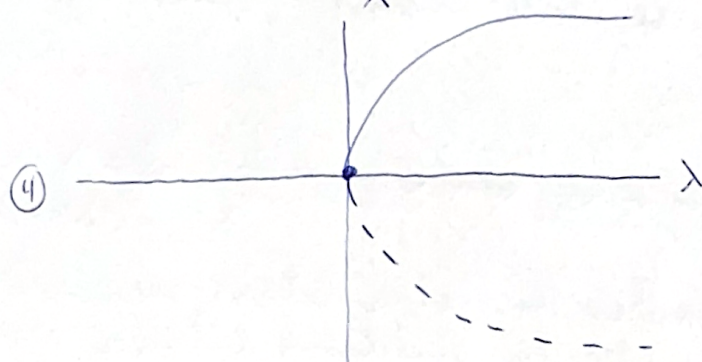
$$D_x f_\lambda(x) \big|_{x=-\sqrt{\lambda}} = +2\sqrt{\lambda} > 0$$

$$D_x f_\lambda(x) \big|_{x=\sqrt{\lambda}} = -2\sqrt{\lambda} < 0$$

So $x = -\sqrt{\lambda}$ is unstable, and $x = \sqrt{\lambda}$ is stable

- ② We may, for each value of λ , plot the position of these fixed points, as well as their stability in the corresponding system on $\mathbb{R}$,

$$\dot{x} = f_{\lambda}(x):$$



Thus, each vertical slice (i.e. fixed value of λ) corresponds to one of the systems in the pictures ①, ② & ③. a solid line means the steady state point is stable in the corresponding ODE $\dot{x} = f_{\lambda}(x)$, a dashed line indicates it is unstable.

The picture ④ is referred to as a bifurcation diagram (of steady state points).

Next, we want to see what happens when we add higher order terms ("h.o.t") to the λ -family of ODEs $\dot{x} = f_{\lambda}(x)$. In other words, we consider

$$\dot{x} = \lambda - x^2 + a_{3,0}x^3 + a_{1,1}x\lambda + a_{0,2}\lambda^2 + \dots$$

for some $a_{ij} \in \mathbb{R}$. (sorta speak)

To do so, we first need some notation and a very useful theorem:

Thm: Implicit function theorem

Let $F: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$ be C^1 (continuously differentiable).

We write $F(x, y)$ with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Suppose $F(a, b) = 0$ and that

$$D_x F(a, b) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a, b) & \dots & \frac{\partial f_m}{\partial x_1}(a, b) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a, b) & \dots & \frac{\partial f_m}{\partial x_m}(a, b) \end{pmatrix} \text{ is invertible}$$

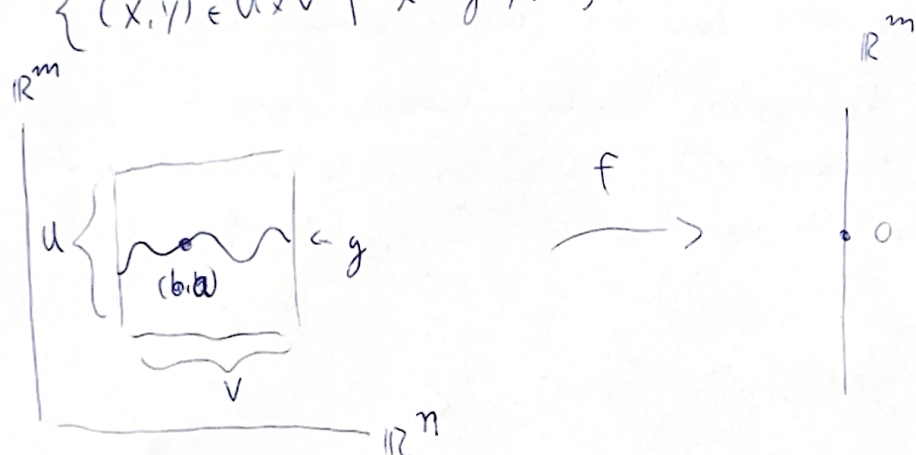
③ Then there exist open neighborhoods $U \subseteq \mathbb{R}^m$ s.t. $a \in U$,
 $V \subseteq \mathbb{R}^n$ s.t. $b \in V$.

and a C^1 map $g: V \rightarrow U$ such that, as sets,

$$\{(x, y) \in U \times V \mid f(x, y) = 0\}$$

"

$$\{(x, y) \in U \times V \mid x = g(y)\}$$



Note that, since $f(a, b) = 0$, we have $a = g(b)$.

• If f is C^k , for $k \in \{1, 2, 3, \dots\} \cup \{\infty\}$, then g can also be assumed C^k .

This theorem follows directly from:

The Inverse function theorem

Let $F: \mathbb{R}^l \rightarrow \mathbb{R}^l$ be C^1 and let $a \in \mathbb{R}^l$ be a point where

$DF(a)$ is invertible (as linear map from $\mathbb{R}^l$ to $\mathbb{R}^l$)

then there exist $U, V \subseteq \mathbb{R}^l$ open, such that $a \in U$ and

$F(U) \subseteq V$ and $H: V \rightarrow U$ a C^1 map such that

$$H(F(x)) = x \text{ for all } x \in U \text{ and}$$

$$F(H(y)) = y \text{ for all } y \in V.$$

So $F|_U$ is a bijection from U to V with C^1 inverse H

• If F is C^k , $k \in \{1, 2, \dots\} \cup \{\infty\}$, then H may be assumed C^k as well

The last part of the Inverse Function Theorem follows readily

④ Because, since $F(H(y)) = y$ for all $y \in V$, we have, by the chain rule.

$$DF(H(y)) \cdot DH(y) = Id. \quad \text{or, } \text{~~setting~~ } \text{~~the~~}$$

$$\text{So } DH(y) = (DF(H(y)))^{-1} = (\mathcal{I} \circ DF \circ H)(y)$$

where $\mathcal{I}: GL(l; \mathbb{R}) \rightarrow$ is the C^∞ map that sends a matrix to its inverse

The Inverse Function Theorem is usually proven by finding a contraction on a suitable function space.

To obtain the Implicit Function Theorem from the Inverse Function Theorem, take $f: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$ as in the Implicit function theorem and define $F: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ as $F(x, y) = (f(x, y), y)$

$$\text{Then } DF(a, b) = \begin{pmatrix} D_x f(a, b) & D_y f(a, b) \\ 0 & Id|_{\mathbb{R}^n} \end{pmatrix}$$

So F satisfies the conditions of the Inverse function theorem $\mathbb{R}^m \rightarrow \mathbb{R}^n$
Let H be the local inverse, and write $H(x, y) = (H_1(x, y), H_2(x, y))$

$$\text{Then } F(H_1(x, y), H_2(x, y)) = (f(H_1(x, y), H_2(x, y)), H_2(x, y)) = (x, y)$$

$$\text{So } H_2(x, y) = y \text{ \& } f(H_1(x, y), y) = x$$

$$\text{Now note that } \{(x, y) \mid f(x, y) = 0\} = \{(x, y) \mid F(x, y) = (0, y)\} \\ = \{H(0, y)\} = \{H_1(0, y), y\}$$

Set $g(y) := H_1(0, y)$ and we are done.

Next, we will often use "big $\mathcal{O}$ " notation. Let $a \in \mathbb{R}^n$ and

$f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ two functions. We write $f(x) = \mathcal{O}(g(x))$ as $x \rightarrow a$
If there exist $C, \delta > 0$ such that

$$|f(x)| \leq C|g(x)| \text{ for all } x \text{ s.t. } 0 < |x - a| < \delta$$

Most of the time, $g(x)$ will be something like $g(x) = |x|^3$

$$\text{or } g(x) = |x|^3 + |x||x| + |x|^2$$

⑤ For instance, suppose $f(x, y) = x^2 f_1(x, y) + y^2 f_2(x, y)$ for some continuous functions $f_1, f_2: \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose $|f_1|$ and $|f_2|$ are bounded by $K > 0$ in some neighbourhood U of $(0, 0) \in \mathbb{R}^2$

then
$$|x^2 f_1(x, y) + y^2 f_2(x, y)| \leq |x^2| |f_1(x, y)| + |y^2| |f_2(x, y)| \leq K(|x|^2 + |y|^2) \quad \text{in } U$$

So $f(x, y) = \mathcal{O}(|x|^2 + |y|^2)$ as $x \rightarrow 0$.

Be careful with the "1.1". For instance $x - y = \mathcal{O}(|x| + |y|)$ as $x \rightarrow 0$

as $|x - y| \leq |x| + |y|$, But $x - y \neq \mathcal{O}(x + y)$, Because for $(x, y) = (\epsilon, -\epsilon)$ as $x \rightarrow 0$

$$|x - y| = 2|\epsilon| \not\leq C|x + y| = 0 \quad \text{in any neighborhood of } (0, 0)$$

$(x, y) = (\epsilon, -\epsilon) \quad (x, y) = (\epsilon, -\epsilon)$

Recall Taylor's theorem:

If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is C^{k+1} , $k \geq 1$, then, for $a \in \mathbb{R}^n$,

$$f(x) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(a)}{\alpha!} (x - a)^\alpha + \sum_{|\beta| = k+1} R_\beta(x) (x - a)^\beta$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices, and every R_β is continuous (so locally bounded)

So then $f(x) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(a)}{\alpha!} (x - a)^\alpha + \mathcal{O}(\sum_{|\beta| = k+1} |x - a|^\beta)$ as $x \rightarrow a$.

Typically, we will want to write $f(x) = \mathcal{O}(g(x))$ as $x \rightarrow 0$

So if we just write $f(x) = \mathcal{O}(g(x))$, we typically mean as $x \rightarrow 0$

Let us now study bifurcations in systems of the form

$$\dot{x} = \lambda - x^2 + \mathcal{O}(|x|^3 + |x||\lambda| + |\lambda|^2) = f(x, \lambda)$$

(say the full system is C^∞ , for simplicity)

- ⑥ Note that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies $D_x f(0,0) = 1$. Thus by the Implicit function, there is a (locally defined) map $\tilde{\lambda}: \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\tilde{\lambda}(0) = 0$ and such that all solutions to $f(x, \lambda) = 0$ are given by $(x, \tilde{\lambda}(x))$. This means the bifurcation diagram may look like, e.g.



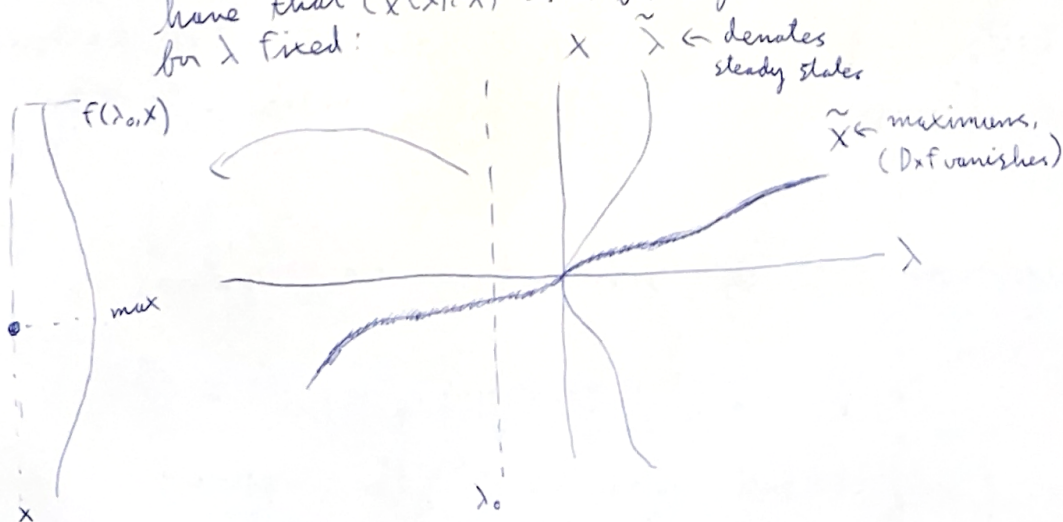
(ignoring stability). All we know is that the λ -components are a function of the x -components, of the steady-state points. Note that the 2nd and 3rd option would constitute "real bifurcations", as the number of zeroes changes as λ -varies.

Now, the derivative in the x -direction of our family of vectorfields is given by $D_x f(x, \lambda) = -2x + \mathcal{O}(|x|^2 + |\lambda|)$. We see that $D_x f(0,0) = 0$ and, since $D_{xx} f(x, \lambda) = -2 + \mathcal{O}(|x| + |\lambda|)$, we conclude again by the Implicit function theorem that, locally,

$\{(x, \lambda) \mid D_x f(x, \lambda) = 0\} = \{(\tilde{x}(\lambda), \lambda)\}$ for some map $\tilde{x}$ passing through 0. Since $D_{xx} f(\tilde{x}(\lambda), \lambda) = -2 + \mathcal{O}(|\lambda|)$, we have

$D_{xx} f(\tilde{x}(\lambda), \lambda) + 2 = \mathcal{O}(|\lambda|)$, so locally

$|D_{xx} f(\tilde{x}(\lambda), \lambda) + 2| \leq C|\lambda|$ for some $C > 0$. This means $D_{xx} f$ is negative along the curve $(\tilde{x}(\lambda), \lambda)$ around $(0,0)$. Thus we have that $(\tilde{x}(\lambda), \lambda)$ is a family of maximums of each $f(x, \lambda)$ for λ fixed:

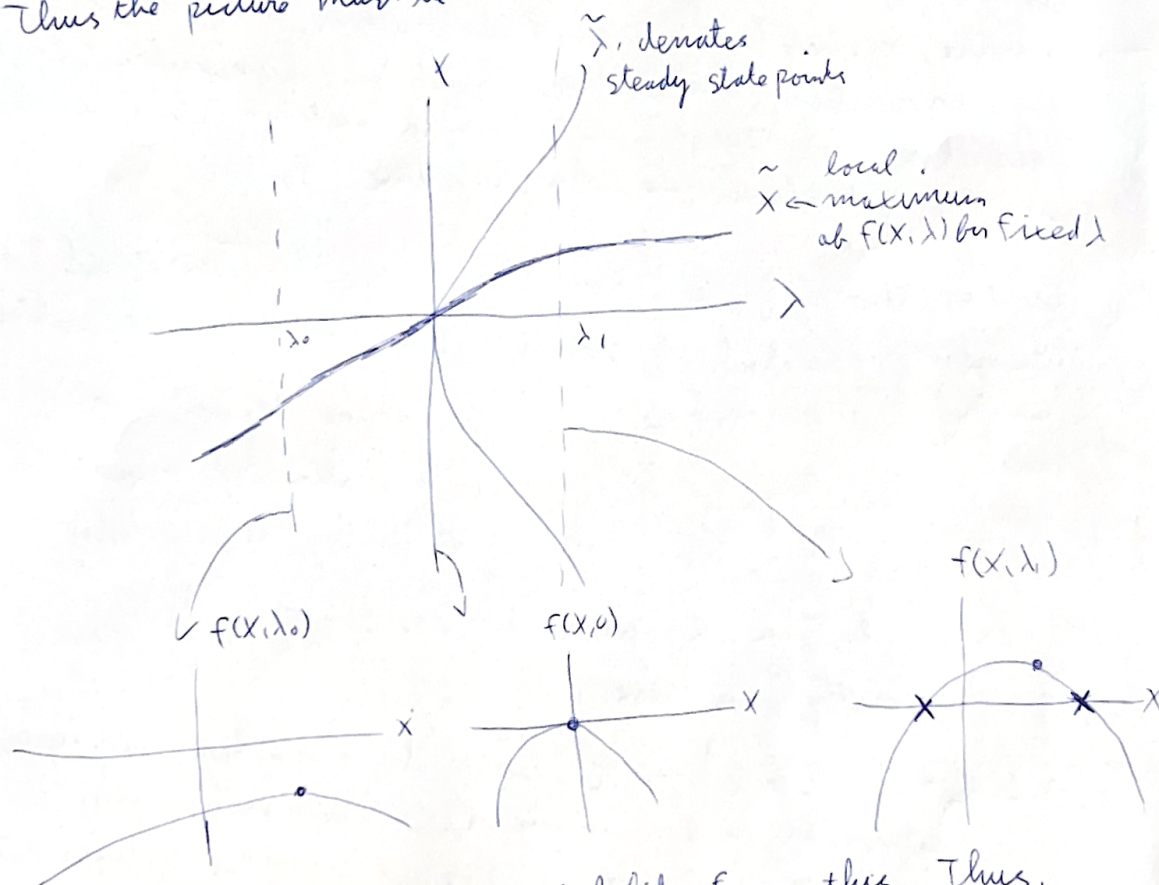


⑦ Finally, consider f along these maximums $\tilde{x}(\lambda)$.

Since $\tilde{x}(\lambda) = \mathcal{O}(|\lambda|)$ and $f(x, \lambda) = \lambda - x^2 + \mathcal{O}(|x|^3 + |x||\lambda| + |\lambda|^2)$, we have

$$f(\tilde{x}(\lambda), \lambda) = \lambda + \mathcal{O}(|\lambda|^2). \quad \text{So for } \lambda < 0, \quad f(\tilde{x}(\lambda), \lambda) < 0 \\ \text{for } \lambda > 0 \quad f(\tilde{x}(\lambda), \lambda) > 0.$$

So for $\lambda < 0$, the maximum of $f(x, \lambda)$ when varying x is negative, so we cannot have any zeroes (locally) for $\lambda < 0$. Thus the picture must be



Note that we may also deduce stability from this. Thus, qualitatively, the bifurcation diagrams for $\dot{x} = \lambda - x^2$ and $\dot{x} = \lambda - x^2 + \mathcal{O}(|x|^3 + |x||\lambda| + |\lambda|^2)$ are the same.

The name for this common steady-state bifurcation is the saddle-node bifurcation.

This is what the vector field $\dot{x} = \lambda - x^2 + \mathcal{O}(|x|^3 + |x||\lambda| + |\lambda|^2)$ means, by the way

$$f(x, \lambda) = \begin{aligned} & a_{00} + a_{10}x + a_{20}x^2 + \overbrace{a_{30}x^3 + \dots}^{\mathcal{O}(|x|^3)} \\ & + \underbrace{a_{01}\lambda + a_{11}x\lambda + a_{21}x^2\lambda + a_{31}x^3\lambda + \dots}_{\mathcal{O}(|x||\lambda|)} \\ & + \underbrace{a_{02}\lambda^2 + a_{12}x\lambda^2 + a_{22}x^2\lambda^2 + a_{32}x^3\lambda^2 + \dots}_{\mathcal{O}(|\lambda|^2)} \\ & + a_{03}\lambda^3 + a_{13}x\lambda^3 + \dots \end{aligned}$$

$$\begin{aligned} \text{So } a_{00} &= 0, \text{ so } f(0, 0) = 0 \\ a_{10} &= 0 \text{ so } D_x f(0, 0) = 0 \\ a_{20} &= -1 \\ a_{01} &= 1 \end{aligned}$$

② It seems a bit excessive that we need 4 conditions for this to hold, but as it turns out this situation is "generic", as we now explore

First, note that, if $f(x_0, \lambda_0) = 0$ for some $x_0, \lambda_0 \in \mathbb{R}$, then we may define $\tilde{f}(x, \lambda) = f(x + x_0, \lambda + \lambda_0)$, so that $\tilde{f}(0, 0) = 0$. Thus, after a quick coordinate transformation, we may assume that the steady-state point that we want to investigate, in terms of continuation, is at $x = 0$ for $\lambda_0 = 0$. This of course works for vectorfields on $\mathbb{R}^n$ in d -parameters as well. One word of advice, going from f to $\tilde{f}$ as above may not preserve all the properties that f has, such as symmetry or a network-structure, at least some care is needed. This explains why in our situation, we may assume $a_{00} = 0$ (i.e. $f(0, 0) = 0$)

As for a_{10} , if $a_{10} \neq 0$, then by the Implicit Function Theorem, the set of steady state points is locally a graph over λ



So there is not really a bifurcation to speak of (of steady state points)

This is why we assume $a_{10} = 0$.

More generally, we have:

Thm 1 Let $F: \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ be a d -parameter family of vectorfields on $\mathbb{R}^n$, which is C^1 (seen as a map from $\mathbb{R}^{n+d} \rightarrow \mathbb{R}^n$)

Suppose $F(0, 0) = 0$, and that $x = 0$ is a hyperbolic steady state point of $F_0 := F(\bullet, 0): \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then there exist open neighborhoods $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^d$ with $0 \in U, 0 \in V$ and a C^1 map $g: V \rightarrow U$ such that:

1) The steady-state points of F are precisely given by $\{(x, \lambda) \in U \times V \mid x = g(\lambda)\}$

2) all the steady state points in $U \times V$ are hyperbolic for their respective systems $F_\lambda := F(\bullet, \lambda)$ on $\mathbb{R}^n$. i.e. each matrix $D_x F(g(\lambda); \lambda)$ is hyperbolic

- ③ The same statement holds true if we replace "hyperbolic" in the theorem above with "invertible", in both instances

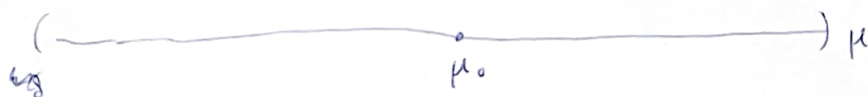
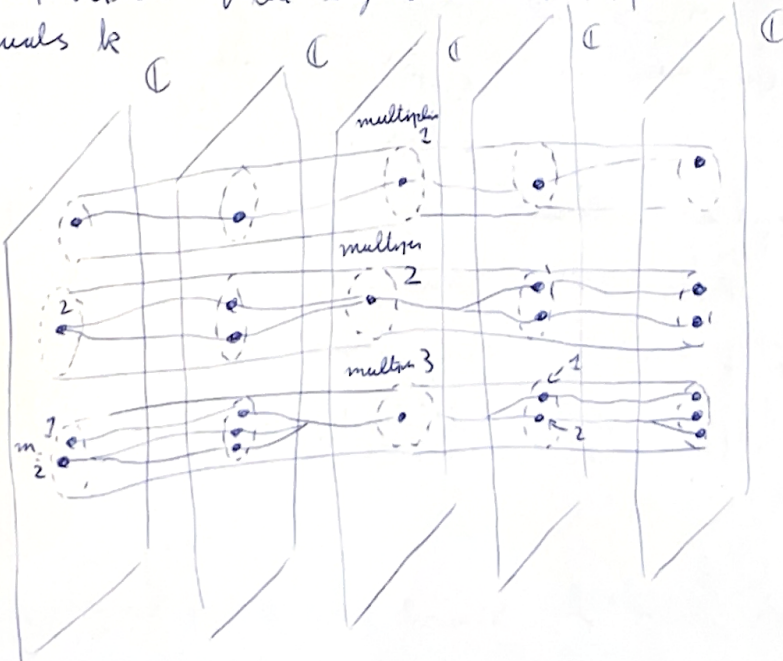
Recall that a matrix is hyperbolic if it has no eigenvalues on the imaginary axis, and invertible if it has no 0-eigenvalues. Thm. 1 essentially tells us that hyperbolic steady-states persist. The following Lemma will be extremely useful:

Lemma 2 Let $A: \mathbb{R}^d \rightarrow \text{Mat}(n, \mathbb{C})$ be a continuous family of complex $n \times n$ matrices. Then the eigenvalues are continuous in the following way: Given μ_0 (and so A_{μ_0}), given an $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|\mu - \mu_0\| < \delta$ then the eigenvalues of A_μ lie in the union of open balls

$B_\varepsilon(\lambda_1) \cup B_\varepsilon(\lambda_2) \cup \dots \cup B_\varepsilon(\lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A_{μ_0} .

In fact: Given μ_0 , given ε and given λ an eigenvalue of A_{μ_0} with algebraic multiplicity k , there is a $\delta > 0$ such that if $\|\mu - \mu_0\| < \delta$, then the sum of the algebraic multiplicities of the eigenvalues of A_μ that lie in $B_\varepsilon(\lambda)$ equals k .

ε small enough
such that
 $B_\varepsilon(\lambda)$
contains
as eigen-
values of
 A_{μ_0} only λ



(10) Proof of Lemma 2: Denote by

$$p_{X, \mu}(X) = X^n + a_{n-1}(\mu)X^{n-1} + a_{n-2}(\mu)X^{n-2} + \dots + a_1(\mu)X + a_0(\mu) \\ = \det(XI_n - A_\mu)$$

Suppose μ_0 , $\varepsilon > 0$ and λ an eigenvalue of A_{μ_0} are given.

Denote by $q(X, \mu) = \frac{\partial p}{\partial X}(X, \mu)$, which is not the 0-polynomial.

Recall ~~that~~ ~~that~~ ~~that~~ that the only zero of $p(X, \mu)$ that is contained in $B_{\varepsilon}(\lambda)$ is λ itself.

Then by the argument principle,

$$\frac{1}{2\pi i} \oint_C \frac{q(z, \mu_0)}{p(z, \mu_0)} dz = \begin{array}{l} \text{"\# zeroes of } p_{\mu_0} \text{ in } B_\varepsilon(\lambda), \\ \text{counted with multiplicity"} \\ = k \end{array}$$

where C denotes a circle of radius ε , integrated counter-clockwise, around λ .

Then C has on it no zeroes of $p(z, \mu_0)$, so (since the circle is compact) there exists an $\delta_1 > 0$ such that $p(z, \mu)$ does not vanish on C if $\|\mu - \mu_0\| < \delta_1$.

So the map $\Psi: B_{\delta_1}(\mu_0) \subseteq \mathbb{R}^d \rightarrow \mathbb{C}$
 $\mu \mapsto \frac{1}{2\pi i} \oint_C \frac{q(z, \mu)}{p(z, \mu)} dz$ is well-defined.

It is also continuous. now choose $\delta_1 > \delta > 0$ such that, if

$$\|\mu - \mu_0\| < \delta \text{ then } \|\Psi(\mu) - k\| < \frac{1}{2}.$$

Since Ψ takes on integer values, necessarily

$$\Psi(\mu) = \frac{1}{2\pi i} \oint_C \frac{q(z, \mu)}{p(z, \mu)} dz = \begin{array}{l} \text{"\# zeroes of } p_\mu \text{ in } B_\varepsilon(\lambda) \\ \text{counted with multiplicity"} \\ = k \end{array}$$

This proves the second part of the Lemma,

The first follows immediately from that

(11) Note that Thm 1 now follows immediately from Lemma 1 and the Implicit Function theorem

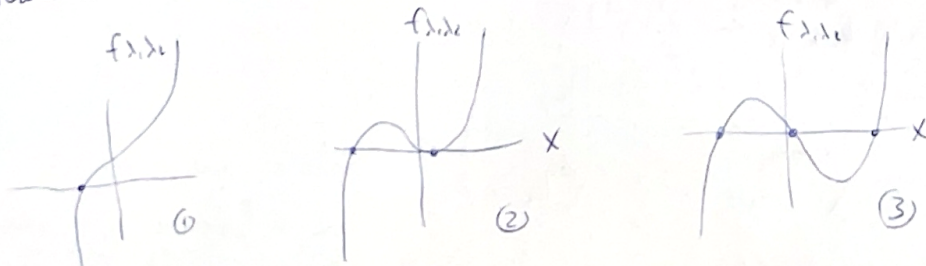
To further drive home that we are not interested in hyperbolic fixed points when doing bifurcation analysis (at least local bifurcation analysis), recall that

Thm Hartman-Grobman: Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 vectorfield and p a hyperbolic fixed point of the flow of F , so $F(p)=0$ and $DF(p) \in \text{Mat}(n; \mathbb{R})$ is hyperbolic. Then there exist open neighborhoods $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi: U \rightarrow V$ which conjugates the restriction of $\dot{x} = F(x)$ on U to that of $\dot{y} = DF(p)y$ on V . (That is, if $\psi_t(x)$ denotes the flow of $\dot{x} = F(x)$ and $\Psi_t(y)$ that of $\dot{y} = DF(p)y$. Then $\phi_t(x)$ is defined iff $\Psi_t(\phi(x))$ is defined, and then $H\phi_t(x) = \Psi_t(H\phi(x))$)

So far we have only considered 1 parameter bifurcations, but others exist as well, of course, consider e.g.

$$\dot{x} = f_{\lambda_1, \lambda_2}(x) = x^3 - \lambda_1 x + \lambda_2, \quad x, \lambda_1, \lambda_2 \in \mathbb{R}$$

Note that, for fixed λ_1, λ_2 , $f_{\lambda_1, \lambda_2}(x) = 0$ has 1, 2 or 3 solutions:



The transition between states (1) and (3) is (2), where we have a double zero. There: $f_{\lambda_1, \lambda_2}(x) = x^3 - \lambda_1 x + \lambda_2 = 0$

$$\text{and } \frac{\partial f_{\lambda_1, \lambda_2}(x)}{\partial x} = 3x^2 - \lambda_1 = 0$$

So $\lambda_1 = 3x^2$, from $\star\star$. Then from $\star$

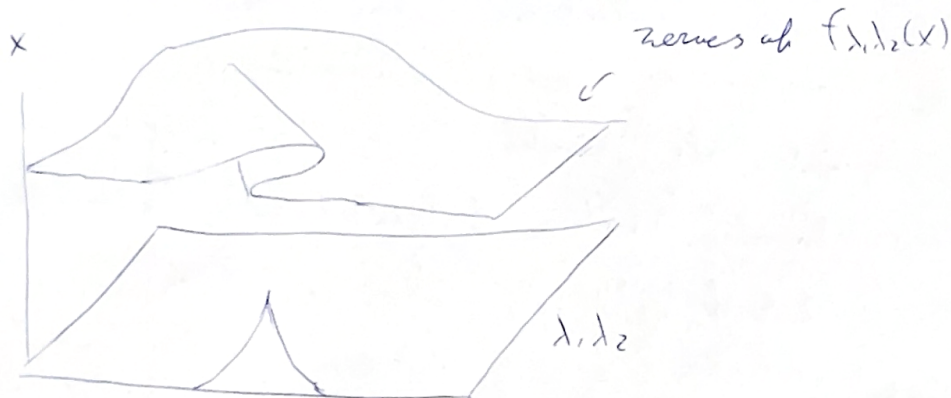
$$0 = x^3 - \lambda_1 x + \lambda_2 = x^3 - 3x^2 \cdot x + \lambda_2 = x^3 - 3x^3 + \lambda_2 = -2x^3 + \lambda_2$$

$$\text{So } \lambda_2 = 2x^3$$

(12) Together, these give

$$\begin{aligned} 4\lambda_1^3 - 27\lambda_2^2 &= 0 \\ 2^2 \cdot 3^3 x^6 - 3^3 2^2 x^6 &= 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} 4\lambda_1^3 - 27\lambda_2^2 &= 0 \\ 2^2 \cdot 3^3 x^6 - 3^3 2^2 x^6 &= 0 \end{aligned}} \right\} \text{So } 4\lambda_1^3 = 27\lambda_2^2$$

This is a cusp, and yields the famous cusp-catastrophe:



Finally, I want to show a bifurcation where the number of steady-state points does not change at all (though the dynamical behavior does change, significantly!).

Consider the system on $\mathbb{R}^2$, parametrised by $\lambda \in \mathbb{R}$;

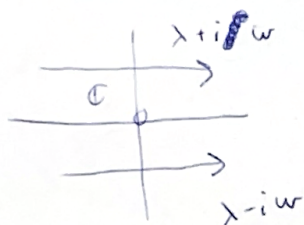
$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} \lambda x - \omega y - (x^2 + y^2)x \\ \omega x + \lambda y - (x^2 + y^2)y \end{pmatrix} \\ &= \begin{pmatrix} \lambda - \omega & 1 \\ \omega & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - (x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where } \omega \in \mathbb{R}_0^+ (\omega > 0) \\ &= F_\lambda(x, y) \quad \text{is a fixed constant (not a variable, nor a bifurcation parameter).} \end{aligned}$$

Note that the eigenvalues of

$$D_x F_\lambda(0,0) = \begin{pmatrix} \lambda - \omega & 1 \\ \omega & \lambda \end{pmatrix} \text{ are given by } \lambda \pm i\omega$$

Thus, $D_x F_\lambda(0,0)$ is always invertible; the isolated steady state point at $(0,0)$ persists. It is in fact the only steady state point of the system, for all values of $\lambda \in \mathbb{R}$.

- (13) However, at $\lambda=0$, the system loses hyperbolicity. In fact, as λ moves from negative to positive, the eigenvalues move as



To see what happens, we write the system as one on ① by setting $z = x + iy$. we get

$$\dot{z} = (\lambda + i\omega)z - |z|^2 z \quad \text{②}$$

This already shows one peculiarity in the system: Suppose $z(t)$ is a solution, and consider the

curve $w(t) = e^{i\alpha} z(t)$, for some $\alpha \in \mathbb{R}$.

$$\begin{aligned} \text{then } \dot{w}(t) &= e^{i\alpha} \dot{z}(t) = e^{i\alpha} (\lambda + i\omega) z(t) - e^{i\alpha} |z(t)|^2 z(t) \\ &= (\lambda + i\omega) e^{i\alpha} z(t) - z(t) \cdot \overline{z(t)} \cdot e^{i\alpha} z(t) \\ &= (\lambda + i\omega) e^{i\alpha} z(t) - (e^{i\alpha} z(t)) \cdot \overline{(e^{i\alpha} z(t))} \cdot e^{i\alpha} z(t) \end{aligned}$$

as $e^{i\alpha} \cdot e^{-i\alpha} = 1$

$$= (\lambda + i\omega) w(t) - |w(t)|^2 w(t)$$

Thus $w(t) := e^{i\alpha} z(t)$ satisfies ② whenever $z(t)$ does. The family of linear maps $\{e^{i\alpha} | \alpha \in \mathbb{R}\}$ "represents the Lie Group S^1 " (More on this later), and we say that the Equation ② has a symmetry: There are linear maps

sending solutions to solutions



Let us now write a solution of ② in polar-coordinates:

$$z(t) = r(t) e^{i\theta(t)}$$

$z(t) \equiv 0$ is always a solution, so if we choose any other solution, this is OK.

(14) Differentiating, we get

$$\begin{aligned}\dot{z} &= \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} \quad \text{so} \\ \bar{z}\dot{z} &= \dot{r}e^{i\theta}re^{-i\theta} + i\dot{\theta}re^{i\theta}re^{-i\theta} \\ &= \dot{r}r + i\dot{\theta}r^2\end{aligned}$$

Thus ~~$\frac{\bar{z}\dot{z}}{r^2}$~~ , ~~$\frac{\bar{z}\dot{z}}{r^2}$~~

$$\begin{aligned}\dot{r} &= \operatorname{Re}\left(\frac{\bar{z}\dot{z}}{r}\right) & \dot{\theta} &= \operatorname{Im}\left(\frac{\bar{z}\dot{z}}{r^2}\right) \\ &= \frac{1}{r}\operatorname{Re}(\bar{z}\dot{z}) & &= \frac{1}{r^2}\operatorname{Im}(\bar{z}\dot{z})\end{aligned}$$

$$\begin{aligned}\text{Now } \bar{z}\dot{z} &= (i\omega + \lambda)z\bar{z} + -|z|^2\dot{z} \\ &= (i\omega + \lambda)|z|^2 - |z|^4 \\ &= (i\omega + \lambda)r^2 - r^4\end{aligned}$$

$$\begin{aligned}\text{Thus } \dot{r} &= \frac{1}{r}\operatorname{Re}((i\omega + \lambda)r^2 - r^4) = \frac{1}{r}(\lambda r^2 - r^4) \\ &= \lambda r - r^3\end{aligned}$$

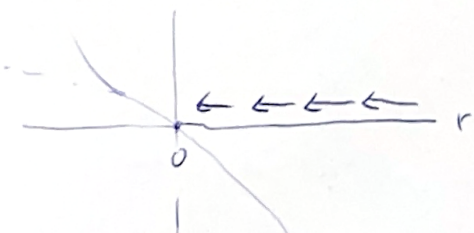
$$\text{and } \dot{\theta} = \frac{1}{r^2}\operatorname{Im}((i\omega + \lambda)r^2 - r^4) = \frac{1}{r^2}\omega r^2 = \omega$$

$$\text{So always } \dot{\theta} = \omega \Rightarrow \theta = \omega t + \theta_0$$

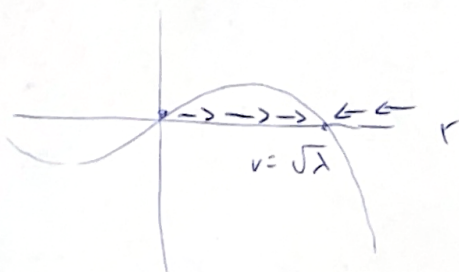
$$\text{and } \dot{r} = \lambda r - r^3$$

Note that $\lambda r - r^3 = 0$ gives $r = 0$ and $r = \pm\sqrt{\lambda}$, if $\lambda \geq 0$.

$$\lambda r - r^3, \lambda \leq 0$$



$$\lambda r - r^3, \lambda > 0.$$



So for $\lambda > 0$, we get the solutions

which "are" $z(t) = \sqrt{\lambda} e^{i(\omega t + \theta_0)}$
periodic (it's one periodic solution, with
differing $z(0)$, of course)

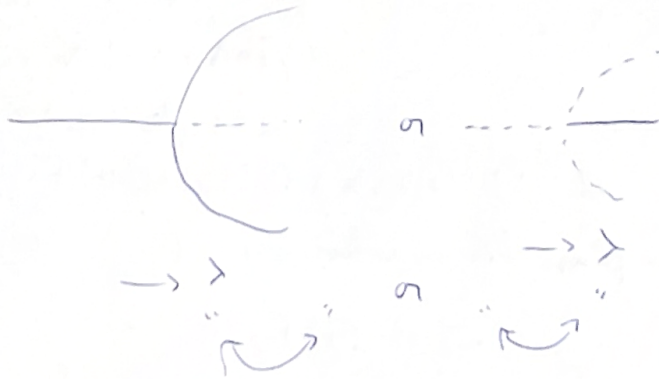
- 15) We get the famous Hopf - Bifurcation for $\lambda \leq 0$, we have a stable steady state point, as $\lambda > 0$, it loses stability, but a stable periodic orbit emerges



Again, the bifurcation persists when higher order terms are added.

- Other "famous bifurcations" to keep an eye on:

- Pitchfork



- Transcritical, i.e.



⑩ Part 2 : Reduction Techniques

We now describe so-called "(complexity) reduction techniques". Most notably dimension reduction techniques. We start with Lyapunov-Schmidt reduction. To illustrate it, consider the 1-parameter family of ODEs on $\mathbb{R}^2$ given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + y + 3\lambda + xy \\ \lambda - xy \end{pmatrix} =: F_\lambda(x, y)$$

Note that $F_0(0,0) = 0$ and $DF_0(0,0) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, the latter of which is not hyperbolic, so we may not assume this steady state point persists. To solve $F_\lambda(x,y) = 0$ near $\lambda=0, x=y=0$,

We need to solve $F'_\lambda(x,y) = -x + y + 3\lambda + xy = 0$ ① &
 $F''_\lambda(x,y) = \lambda - xy = 0$ ②.

From the Implicit Function Theorem, as $D_x F'_\lambda(0,0) = -1 + y|_{y=0} = -1$

We see that ① is locally solved by $x = x(\lambda, y)$

We write $x(\lambda, y) = ay + b\lambda + \mathcal{O}(|y|^2 + |y||\lambda| + |\lambda|^2)$, then plugging into ① we get

$$\begin{aligned} -x(\lambda, y) + y + 3\lambda + x(\lambda, y)y &= 0 \\ -ay - b\lambda + y + 3\lambda + \mathcal{O}(|y|^2 + |y||\lambda| + |\lambda|^2) &= 0 \\ \Rightarrow a=1, b=3, \text{ so} \end{aligned}$$

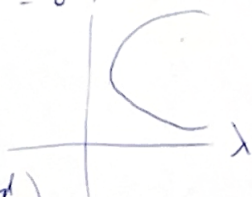
$$x(\lambda, y) = y + 3\lambda + \mathcal{O}(|y|^2 + |y||\lambda| + |\lambda|^2)$$

Now we plug this equation for $x(\lambda, y)$ into ②, to obtain

$$\begin{aligned} \lambda - x(\lambda, y)y &= 0 \Rightarrow \\ \lambda - y^2 - 3\lambda y + \mathcal{O}(|y|^3 + |y||\lambda| + |\lambda|^2) &= 0 \\ \Rightarrow \lambda - y^2 + \mathcal{O}(|y|^3 + |\lambda||y|) &= 0 \end{aligned}$$

So the steady states look like

(with the equations for x determined by λ and y)



(7) So what did we do here? This is an instance of so-called Lyapunov-Schmidt Reduction. Suppose we have a family of vectorfields $(\infty)F: \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ s.t. $F(0;0)=0$. To solve for $F(x;\lambda)=0$ locally around $(x;\lambda)=(0,0)$, let us write $A := D_x F(0;0)$. We choose a vector space $U \subseteq \mathbb{R}^n$ complementary to $\ker(A)$ and a vector space $V \subseteq \mathbb{R}^n$ complementary to $\text{Im}(A)$. Thus

$$\mathbb{R}^n = U \oplus \ker(A) = V \oplus \text{Im}(A). \text{ Denote by}$$

$P_V: \mathbb{R}^n \rightarrow V$ the projection onto V along $\text{Im}(A)$. So

$\text{Id}_V - P_V: \mathbb{R}^n \rightarrow \text{Im}(A)$ is the projection onto $\text{Im}(A)$ along V .

Then $F(x;\lambda)=0$ is equivalent to

$$(\text{Id}_V - P_V)F(x;\lambda) = (\text{Id}_V - P_V)F(u+y;\lambda) = 0 \quad (1)$$

&

$$P_V F(x;\lambda) = P_V F(u+y;\lambda) = 0 \quad (2)$$

where we write $x = u + y$ with $u \in U$, $y \in \ker(A)$.

We first focus on Equation (1): Define $G: U \times \ker(A) \times \mathbb{R}^d \rightarrow \text{Im}(A)$

$$\text{by } G(u, y; \lambda) = (\text{Id}_V - P_V)(F(u+y; \lambda))$$

$$\text{given } p \in U, \text{ we have } D_u G(0,0;0)p = \frac{d}{dt} \big|_{t=0} G(tp, 0; 0)$$

$$= \frac{d}{dt} \big|_{t=0} (\text{Id}_V - P_V) F(tp; 0) = (\text{Id}_V - P_V) D_x F(0;0)p = (\text{Id}_V - P_V) A p = A p.$$

where in the last step we use that $A p \in \text{Im}(A)$, so $(\text{Id}_V - P_V) A p = A p$

Now, if $D_u G(0,0;0)p = 0$ then $A p = 0$ so $p \in \ker A$, but $p \in U$ and $\mathbb{R}^n = U \oplus \ker(A)$ so $p = 0$. Thus $D_u G(0,0;0)$ is injective.

$$\text{By rank-nullity} \quad \dim^{\ker}(A) + \dim(\text{Im}(A)) = \dim(\mathbb{R}^n) = n$$

(As $0 \rightarrow \ker(A) \xrightarrow{i} \mathbb{R}^n \xrightarrow{F} \text{Im}(A) \rightarrow 0$ is exact)

Also $\dim(U) + \dim(\ker(A)) = n$ so $\dim(\text{Im}(A)) = \dim(U)$. Thus

$D_u G(0,0;0): U \rightarrow \text{Im}(A)$ is bijective. So by the

Implicit Function theory: (1) is solved locally by $(u(y;\lambda), y; \lambda)$

Now define $H: \ker(A) \times \mathbb{R}^d \rightarrow V$ by

$$H(y; \lambda) = P_V F(u(y; \lambda) + y; \lambda)$$

$$\textcircled{3}: H(y; \lambda) = 0$$

Then solutions to (3) are the same as solutions to (2), given (1) is satisfied.

(10) Note as well that

$$\dim(V) = n - \dim(\text{Im}(A)) = \dim(\ker(A)).$$

So the only place we have to get "our hands dirty" is at solving $H(y; \lambda) = 0$ for $y \in \ker(A)$.

Example

Consider the ODE:

$$\dot{X}_1 = -X_1 + \sin(X_1) y^2$$

$$\dot{X}_2 = -X_2 + 2 \sin(X_2) y^2$$

$$\dot{X}_3 = -X_3 + 3 \sin(X_3) y^2$$

$\vdots$

$$\dot{X}_{100} = -X_{100} + 100 \sin(X_{100}) y^2$$

$$\dot{y} = -\lambda + y^2 + X_1^3 + X_2^3 + \dots + X_{100}^3$$

$$= f_\lambda(X_1, \dots, X_{100}, y)$$

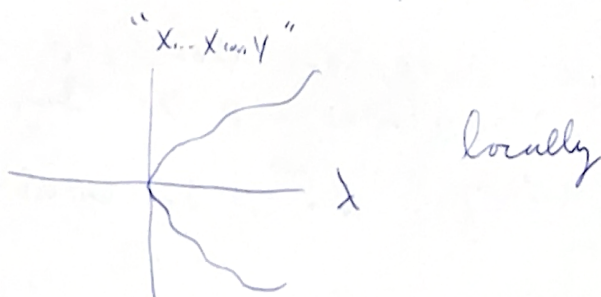
with $X_1, X_2, \dots, X_{100}, y, \lambda \in \mathbb{R}$. What is the structure of the zeroes near $(0, 0, \dots, 0, 0; 0) \in \mathbb{R}^{102}$?

By the Implicit Function Theorem, $\star$ $X_i = X_i(y; \lambda) = \mathcal{O}(|y| + |\lambda|)$
 then to solve for the y -variable:

$$0 = -\lambda + y^2 + X_1^3 + \dots + X_{100}^3 = -\lambda + y^2 + \mathcal{O}(|y|^3 + |\lambda|^2 |y| + |\lambda| |y|^2 + |\lambda|^3)$$

Thus we have to solve $\lambda - y^2 + \mathcal{O}(|y|^3 + |\lambda|(|y| + |\lambda|^2)) = 0$

So again we get



The Lyapunov - Schmidt Reduction technique is more of a "big deal" than you might think: as we have:

Thenam (informally): Let $U \subseteq \mathbb{R}^d$ open, bounded; and let

$C(U; n)$ denote all smooth maps $l: U \rightarrow \text{Mat}(\mathbb{R}; n)$

Then for "generic choice" of $l \in C(U; n)$, we have

$$\dim(\ker l(x)) \leq d \text{ for all } x \in U$$

①9 Thus, in particular, along a 1-parameter bifurcation (so $d=1$) we have generically that $\dim(\ker(A))=1$. So then the reduced equation $H(y;\lambda)$ just becomes a problem on $\mathbb{R}$ (with 1-parameter) (even if $n=10^7$!) More on this later. Later, I also want to show Lyapunov-Schmidt Reduction on Banach spaces.

Note that one problem with Lyapunov-Schmidt reduction is that we do not (easily) get information on stability. This is different with our next technique:

Center Manifold Reduction

To motivate it, we need the following result & definition:

Lemma Let $A \in \text{Mat}(\mathbb{R}; n)$ be a given real $n \times n$ matrix, and denote by $\Delta_1, \Delta_2, \dots, \Delta_l \subseteq \mathbb{R}$ a partition of $\mathbb{R}$ into l parts. So $\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j$ ($i, j \in \{1, \dots, l\}$) and $\mathbb{R} = \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_l$. Then there exist unique linear subspaces $U_1, U_2, \dots, U_l \subseteq \mathbb{R}^n$ s.t.

- 1) $A U_i \subseteq U_i$ for all $i \in \{1, \dots, l\}$
- 2) $A|_{U_i} : U_i \rightarrow U_i$ has only (generalised) eigenvalues with real parts in Δ_i , for all $i \in \{1, \dots, l\}$
- 3) $U_1 \oplus U_2 \oplus \dots \oplus U_l = \mathbb{R}^n$

Proof This follows directly from real Jordan-Normal form. Note that, if λ is an eigenvalue of A with $\text{Re}(\lambda) \in \Delta_i$ then $\bar{\lambda}$ is an eigenvalue, satisfying likewise $\text{Re}(\bar{\lambda}) \in \Delta_i$ (as $\text{Re}(\lambda) = \text{Re}(\bar{\lambda})$)

Definition Two cases are important to us:

Case 1) $\Delta_1 = \{x \in \mathbb{R} \mid x < 0\}$, $\Delta_2 = \{0\}$, $\Delta_3 = \{x \in \mathbb{R} \mid x > 0\}$

Then U_1 is called the stable subspace of A , we write $U_1 = U_s$
 U_2 is called the center subspace of A , we write $U_2 = U_c$
 U_3 is called the unstable subspace of A , we write $U_3 = U_u$

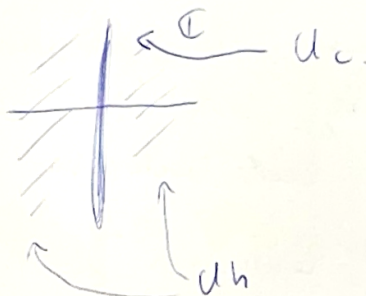
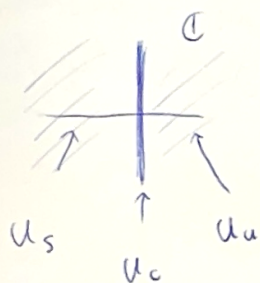
(20)

Case 2) $\Delta_1 = \{x \in \mathbb{R} \mid x \neq 0\}$, $\Delta_2 = \{0\}$ then

U_1 is the hyperbolic subspace of A , $U_1 = U_h$

U_2 is the center subspace of A , $U_2 = U_c$ (as before).

Note that $U_h = U_s \oplus U_u$, and so $\mathbb{R}^n = U_h \oplus U_c = U_s \oplus U_u \oplus U_c$

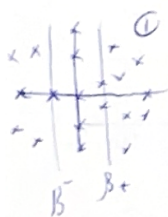


(Of course, U_c is the same in both definitions)

We note the following: dynamics on U_u go to infinity as $t \rightarrow \infty$ while those for U_s go to infinity as $t \rightarrow -\infty$. That is, dynamics of the system $\dot{x} = Ax$, so $e^{tA}x_0$ for $x_0 \in U_u, U_s$ respectively. So the bounded solutions, if present, have to lie on U_c . Of course these are not too interesting for linear systems, but we can have families of steady state points or periodic orbits, and these have to lie on U_c . We explore dynamics of $\dot{x} = Ax$ some more in the following:

Prop 1: Denote by π_c & $\pi_h = \text{Id} - \pi_c$ the projections along $\mathbb{R}^n = U_h \oplus U_c$ onto U_c and U_h , resp.

Similarly, define π_c, π_u, π_s w.r.t. $\mathbb{R}^n = U_c \oplus U_u \oplus U_s$



Set $\beta_+ := \min \{ \text{Re}(\lambda) \mid \lambda \text{ eigenvalue of } A \text{ and } \lambda > 0 \}$

$\beta_- := \max \{ \text{Re}(\lambda) \mid \lambda \text{ eigenvalue of } A \text{ and } \lambda < 0 \}$

Then given $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ s.t.

$$\| e^{A+t} \pi_c \| \leq M(\varepsilon) e^{\varepsilon|t|} \quad \forall t \in \mathbb{R}$$

$$\| e^{A+t} \pi_u \| \leq M(\varepsilon) e^{(\beta_+ - \varepsilon)t} \quad \forall t \leq 0$$

$$\| e^{A+t} \pi_s \| \leq M(\varepsilon) e^{(\beta_- + \varepsilon)t} \quad \forall t \geq 0$$

Thus the U_c part may still grow, but less than any ^{given} exponential.

The stable part will decrease, and do better than "the highest eigenvalue plus a bit"

21) The proof of Prop 1 uses the so-called Jordan - Chevalley decomposition. It states, given $A \in \text{Mat}(n; \mathbb{R})$, there exist unique $S, N \in \text{Mat}(n; \mathbb{R})$, satisfying

- 1) $A = S + N$
- 2) S is semisimple (diagonalizable in $\text{Mat}(n; \mathbb{C})$)
- 3) N is nilpotent ($N^k = 0$ for some $k > 0$)
- 4) $SN = NS$

Note that, if A has the Jordan - normal form

$$BAB^{-1} = \begin{pmatrix} \lambda_1 & 1 & 0 & \dots & 0 \\ & \lambda_1 & 1 & \dots & 0 \\ & & \ddots & \ddots & \\ & & & \lambda_s & 1 \\ 0 & & & & \lambda_s \end{pmatrix} \quad \text{then } BSB^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ & \lambda_1 & 0 & \dots & 0 \\ & 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \ddots & \\ & 0 & & \lambda_s & 0 \\ 0 & & & & \lambda_s \end{pmatrix}$$

$$\text{and } BNB^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \dots & 0 \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{pmatrix}$$

This is useful for intuition and makes the statement believable (in the absence of a proof)

We also point out that $\pi_{c/u/s/h} \circ A = A \circ \pi_{c/u/s/h}$ by definition, and so likewise, as $e^{tA} = \text{Id} + tA + \frac{t^2 A^2}{2!} + \dots$, we have

$$\pi_{c/u/s/h} \circ e^{tA} = e^{tA} \pi_{c/u/s/h}. \quad \text{Thus in Prop 1, we will see}$$

$e^{At} \pi_c$ as a linear map from U_c to U_c , and likewise for $e^{At} \pi_s$ and $e^{At} \pi_u$ from $U_s \rightarrow U_s$, $U_u \rightarrow U_u$ resp.

as all norms on $\mathbb{R}^n$ are equivalent, we get the statement for $e^{At} \pi_{c/u/s} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by setting $\|x\| := \|x_c\|_c + \|x_s\|_s + \|x_u\|_u$, when

$$x = x_c + x_s + x_u, \text{ with } x_c \in U_c, x_s \in U_s, x_u \in U_u$$

and with our favourite norms $\|\cdot\|_c, \|\cdot\|_s$ and $\|\cdot\|_u$ on U_c, U_s and U_u respectively

as we will see in the proof of Prop. 1, the ε 's come from the nilpotent parts. e.g. let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$\text{So the ODE } \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ looks like } \dot{x} = y, \dot{y} = z, \dot{z} = 0$$

(22) So $z(t) = z_0$, $y(t) = z_0 t + y_0$, $x(t) = \frac{z_0}{2} t^2 + y_0 t + x_0$.

This can have ~~exponential~~ growth, e.g. ~~if~~ if $x_0, y_0, z_0 > 0$. But it is polynomial, not exponential.

Proof of Prop 7 We fix $\varepsilon > 0$ and write $A_c := \pi_c A \pi_c : U_c \rightarrow U_c$ then we may see $e^{tA} \pi_c$ as a map from U_c to U_c which is given by e^{tA_c} . Write $A_c = S_c + N_c$ in Jordan Chevalley Decomposition. As $S_c \circ N_c = N_c \circ S_c$, we have

$$e^{tA_c} = e^{tN_c} \cdot e^{tS_c}, \text{ so } \|e^{tA_c}\| \leq \|e^{tN_c}\| \cdot \|e^{tS_c}\|.$$

As S_c is semisimple with all eigenvalues on $i\mathbb{R}$, so

$$S_c \sim \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & \begin{bmatrix} 0-w_1 & \\ & w_1 0 \end{bmatrix} & \\ & & & \begin{bmatrix} 0-w_r & \\ & w_r 0 \end{bmatrix} \end{pmatrix}$$

we have $\|e^{tA_c}\| \leq C$ for some $C > 0$

More precise, if D is diagonal (or 2×2 block diagonal) then $\|D\| \leq 1$ and we

may write $A_c = BDB^{-1}$, so $e^{tA_c} = e^{tBDB^{-1}} = B e^{tD} B^{-1}$, so

$$\|e^{tA_c}\| \leq \|B\| \|B^{-1}\| \|e^{tD}\| \leq \|B\| \|B^{-1}\| =: C$$

Now we look at e^{tN_c} . Assume $N_c^{k+1} = 0$ for $k \geq 0$. Then

$$e^{tN_c} = \text{Id} + tN_c + \frac{t^2}{2!} N_c^2 + \dots + \frac{t^k}{k!} N_c^k \quad \text{So}$$

$$\begin{aligned} \|e^{tN_c}\| &\leq \|\text{Id}\| + |t| \|N_c\| + \frac{|t|^2}{2!} \|N_c\|^2 + \dots + \frac{|t|^k}{k!} \|N_c\|^k \\ &= 1 + |t| \|N_c\| + \dots + \frac{|t|^k}{k!} \|N_c\|^k \end{aligned}$$

$$\begin{aligned} &= 1 + |t| \left(\frac{\|N_c\|}{\varepsilon} \right) + \frac{|t|^2}{2!} \frac{\|N_c\|^2}{\varepsilon^2} + \dots + \frac{|t|^k}{k!} \frac{\|N_c\|^k}{\varepsilon^k} \\ &\leq \tilde{C}(\varepsilon) \left(1 + |t| + \frac{|t|^2}{2!} + \dots + \frac{|t|^k}{k!} \right) \end{aligned}$$

$$\text{where } \tilde{C}(\varepsilon) = \max \left(1, \frac{\|N_c\|}{\varepsilon}, \frac{\|N_c\|^2}{\varepsilon^2}, \dots, \frac{\|N_c\|^k}{\varepsilon^k} \right) > 0$$

$$\begin{aligned} \text{So } \|e^{tN_c}\| &\leq \tilde{C}(\varepsilon) \left(1 + |t| + \dots + \frac{|t|^k}{k!} \right) \\ &\leq \tilde{C}(\varepsilon) \left(1 + |t| + \frac{|t|^2}{2!} + \dots + \frac{|t|^k}{k!} + \frac{|t|^{k+1}}{(k+1)!} + \dots \right) \\ &= \tilde{C}(\varepsilon) e^{|t|} = \tilde{C}(\varepsilon) e^{\varepsilon |t|} \end{aligned}$$

Thus putting things together: $\|e^{tA_c}\| \leq \|e^{tN_c}\| \cdot \|e^{tS_c}\|$

$$\begin{aligned} &\leq C \cdot \tilde{C}(\varepsilon) e^{\varepsilon |t|} \\ &= M(\varepsilon) e^{\varepsilon |t|} \quad \text{if } M(\varepsilon) := C \cdot \tilde{C}(\varepsilon) \end{aligned}$$

(23) The cases $\|e^{A^t} \pi_s\|$ and $\|e^{A^t} \pi_u\|$ go similarly. Denote by A_s the restriction of A to U_s , and let $A_s = N_s + S_s$ be its Jordan Chevalley decomposition. Then there are invertible $B: U_s \rightarrow U_s$ and a matrix

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_2 & \\ 0 & & & \begin{pmatrix} \lambda_s & w_s \\ -w_s & \lambda_s \end{pmatrix} \end{pmatrix}$$

with all $\lambda_i < 0$ and $\beta_- = \max\{\lambda_i\}$

such that $S_s = B D B^{-1}$

and for $e^{t S_s} = B e^{t D} B^{-1}$

From this $\|e^{t S_s}\| \leq \|B\| \|B^{-1}\| \|e^{t D}\| = \|B\| \|B^{-1}\| e^{t \beta_-}$ for $t \geq 0$

also as before $\|e^{t N_s}\| \leq \tilde{C}(\epsilon) e^{\epsilon |t|}$ for all t , so $\|e^{t N_s}\| \leq \tilde{C}(\epsilon) e^{\epsilon t}$

and thus $\|e^{t A_s}\| \leq \|e^{t N_s}\| \|e^{t S_s}\| \leq M(\epsilon) e^{t(\beta_- + \epsilon)}$ for $M(\epsilon) = \|B\| \|B^{-1}\| \tilde{C}(\epsilon)$.

The case $\|e^{t A} \pi_u\|$ is completely analogous, or deduce from the stable case by sending $t \mapsto -t$, $A \mapsto -A$. This completes the proof $\square$

Prop 1 Will be very important later on. For now, we explore a consequence for linear systems:

Lem 2 Let $A \in \text{Mat}(n; \mathbb{R})$ and denote by $\varphi_t(x)$ the flow of the ODE $\dot{x} = Ax$. Let U_c, U_h be as defined before using A , and denote by $\pi_h: \mathbb{R}^n \rightarrow U_h$ the projection onto U_h along U_c . Then we have

$$U_c = \left\{ x \in \mathbb{R}^n \mid \sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty \right\} = \left\{ x \in \mathbb{R}^n \mid \sup_{t \in \mathbb{R}} e^{-\eta |t|} \|\varphi_t(x)\| < \infty \right\}$$

where η is any real number satisfying $0 < \eta < \beta$
with $\beta = \min\{\beta_+, -\beta_-\}$



(So all eigenvalues λ of A are either on $i\mathbb{R}$, or satisfy $|\text{Re}(\lambda)| > \eta$)

(24) Proof of Lemma 2 • If $x \in U_c$ then $\varphi_t(x) = e^{tA}x \in U_c$ for all $t \in \mathbb{R}$,

$$\text{so } \sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| = \sup_{t \in \mathbb{R}} \|0\| = 0 < \infty.$$

• Now suppose $x \in \mathbb{R}^n$ is such that $\sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty$, say $\sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| = C$.

then we may write $x = x_c + x_h$ for $x_h = \pi_h(x) \in U_h$, $x_c \in U_c$.

$$\begin{aligned} \text{We get } \|\varphi_t(x)\| &= \|e^{tA}(x_c + x_h)\| = \|e^{tA}x_c + e^{tA}x_h\| \leq \|e^{tA}x_c\| + \|e^{tA}x_h\| \\ &= \|e^{tA}x_c\| + \|\cancel{e^{tA}} e^{tA} \pi_h x\| = \|e^{tA}x_c\| + \|\pi_h e^{tA}x\| = \|e^{tA}x_c\| + \|\pi_h \varphi_t(x)\| \\ &\leq M(n) e^{n|t|} + C \quad \text{by Prop 1.} \end{aligned}$$

$$\leq M(n) e^{n|t|} + C e^{n|t|} \quad (\text{as } e^{n|t|} \geq 1 \text{ for all } t \in \mathbb{R})$$

$$\text{So } e^{-n|t|} \|\varphi_t(x)\| \leq M(n) + C \quad \text{thus } \sup_{t \in \mathbb{R}} e^{-n|t|} \|\varphi_t(x)\| < \infty$$

• Now suppose $x \in \mathbb{R}^n$ satisfies

$$\sup_{t \in \mathbb{R}} e^{-n|t|} \|\varphi_t(x)\| < \infty, \text{ so}$$

$$\|\varphi_t(x)\| \leq C e^{n|t|} \text{ for some } C > 0. \text{ Then for } t \leq 0: (\text{by Prop 2})$$

$$\begin{aligned} \|\pi_u x\| &= \|e^{At} \pi_u e^{-At} x\| \leq M(\varepsilon) e^{(B_+ - \varepsilon)t} \|e^{-At} x\| \\ &= M(\varepsilon) e^{(B_+ - \varepsilon)t} \|\varphi_{-t}(x)\| \\ &\leq M(\varepsilon) e^{(B_+ - \varepsilon)t} \cdot C e^{n|t|} \\ &= M(\varepsilon) \cdot C \cdot e^{(B_+ - \varepsilon)t} e^{-\eta t} \quad (\text{as } t \leq 0, \text{ so } |t| = -t) \\ &= M(\varepsilon) \cdot C \cdot e^{(B_+ - \eta - \varepsilon)t} \end{aligned}$$

for any $\varepsilon > 0$. $\rightarrow$

So choose ε such that

$$\varepsilon < B_+ - \eta \leq B_+ - \eta \quad \text{so } (B_+ - \eta - \varepsilon) > 0$$

$$\text{Thus } \|\pi_u x\| \leq M(\varepsilon) \cdot C \cdot e^{(B_+ - \eta - \varepsilon)t} \text{ for all } t \leq 0. \text{ So let } t \rightarrow -\infty$$

$$\text{and we get } \|\pi_u x\| \leq 0 \Rightarrow \pi_u x = 0.$$

Similarly, for $t \geq 0$ and $\varepsilon < -B_- - \eta$, so $B_- + \eta + \varepsilon < 0$:

$$\|\pi_s x\| \leq M(\varepsilon) e^{(B_- + \varepsilon)t} \cdot C e^{n|t|} = M(\varepsilon) C e^{(B_- + \eta + \varepsilon)t}$$

$$\text{as } t \rightarrow \infty: \|\pi_s x\| \leq 0 \Rightarrow \pi_s x = 0$$



$$\text{So } x = x_c + \pi_u x + \pi_s x = x_c + 0 + 0 = x_c \in U_c$$

This completes the proof. Note that Lem 2

Confirms our previous statement that all bounded solutions lie on U_c , as $\varphi_t(x)$ bounded

implies $\pi_h \varphi_t(x)$ bounded ($\|\pi_h \varphi_t(x)\| \leq \|\pi_h\| \|\varphi_t(x)\|$) and so $x \in U_c$

$$\begin{aligned} -\eta &\geq B_- \\ \text{so } B_- + \eta &\leq 0 \\ B_- + \eta + \varepsilon &< 0 \end{aligned}$$

(25) We may now finally state our main result. It says that for systems "sufficiently close to linear" (but still with nonlinear terms) we have an invariant manifold closely mirroring the space U_c and its properties as laid out in Lemma 2:

Thm 3 [Center Manifold Reduction] Given $A \in \text{Mat}(n, \mathbb{R})$, denote by W_c and W_h its center and hyperbolic subspaces (named U_c & U_h before). Let $k \geq 1$ be given. There exists $\varepsilon = \varepsilon(A, k) > 0$ such that if a non-linearity $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $G(0) = 0$, $DG(0) = 0$ and

$$1) \quad \sup_{x \in \mathbb{R}^n} \|D^j G(x)\| < \infty \quad \text{for } 0 \leq j \leq k$$

$$2) \quad \sup_{x \in \mathbb{R}^n} \|DG(x)\| < \varepsilon$$

Then the system $\dot{x} = Ax + G(x)$ has the following property:

There exists a function $\Psi: W_c \rightarrow W_h$ of class C^k s.t. its flow graph $\{x_c + \Psi(x_c) \mid x_c \in W_c\}$ is flow-invariant. Ψ moreover is bounded and satisfies $\Psi(0) = 0$ and $D\Psi(0) = 0$. We usually refer to this flow-invariant set as the Center Manifold of the system:

$$M_c = \{x_c + \Psi(x_c) \mid x_c \in W_c\}$$

We moreover have

$$M_c = \{x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty\}$$

where $\varphi_t(x)$ denotes the flow of the system $\dot{x} = Ax + G(x)$ and $\pi_h: \mathbb{R}^n \rightarrow W_h$ is the projection w.r.t. $\mathbb{R}^n = W_c \oplus W_h$. Also:

$$M_c = \left\{ x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|\varphi_t(x)\| < \infty \right\} \quad \left(\begin{array}{l} \text{with } \eta \in (0, \beta) \\ \text{with } \beta \text{ defined by } A \\ \text{as before} \end{array} \right)$$

Finally: if $\phi: W_c \rightarrow W_h$ is continuous, bounded

and such that the set $\{x_c + \phi(x_c) \mid x_c \in W_c\}$

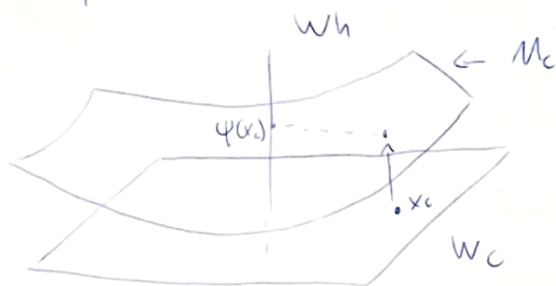
is flow-invariant, then $\phi = \Psi$

Note that, if $G \equiv 0$ (so if the ODE is $\dot{x} = Ax$, i.e. linear) then 1) & 2) are satisfied and $\Psi \equiv 0$, so $M_c = W_c$

and we retrieve Lemma 2.

by uniqueness

(26) In general, the picture is:



Note that the set: $\{x \in \mathbb{R}^n \mid \sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty\}$ is flow-invariant by construction. For, if $\|\pi_h \varphi_t(x)\| < C$ for some $C > 0$ and all $t \in \mathbb{R}$, then given $t_0 \in \mathbb{R}$, $\|\pi_h \varphi_t(\varphi_{t_0}(x))\| = \|\pi_h \varphi_{t+t_0}(x)\| < C$ for all $t \in \mathbb{R}$. (Note as well that, since G is assumed bounded, solutions to $\dot{x} = Ax + G(x)$ exist for all $t \in \mathbb{R}$!)

If $\varphi_t(x)$ is a bounded solution, say $\|\varphi_t(x)\| < D$ for some $D > 0$ and all $t \in \mathbb{R}$, then clearly $\|\pi_h \varphi_t(x)\| \leq \|\pi_h\| \|\varphi_t(x)\| \leq \|\pi_h\| D < \infty$ so M_c contains all bounded solutions. However, M_c can (and will often) contain unbounded solutions too. Consider for instance the linear system $\dot{x} = Ax$ (so $G \equiv 0$, so $M_c = W_c$) where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and so $M_c = \mathbb{R}^2$. we already say that $\dot{x} = Ax$ ($\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$) has unbounded solutions $y(t) = y_0 \neq 0$
 $x(t) = y_0 t + x_0$

Some more remarks on the theorem: the system $\dot{x} = F(x) := Ax + G(x)$ has a steady state point in 0: $F(0) = G(0) = 0$, so $0 \in M_c$, and so $\psi(0) = 0$ necessarily. Thus that ballows

Regarding the uniqueness: if $\phi: W_c \rightarrow W_h$ is continuous and bounded, then say $\|\phi(x_c)\| < C$ for some $C > 0$ and all $x_c \in W_c$. If in addition $\{x_c + \phi(x_c) \mid x_c \in W_c\}$ is flow invariant, then necessarily for $x_0 = x_c + \phi(x_c)$, we have $\pi_h(\varphi_t(x_0)) = \phi(\pi_c(\varphi_t(x_0)))$ so $\|\pi_h(\varphi_t(x_0))\| < C$.

So $\|\pi_h \varphi_t(x_0)\|$ is bounded

and thus $x_0 = x_c + \phi(x_c) \in \{x_c + \psi(x_c) \mid x_c \in W_c\}$

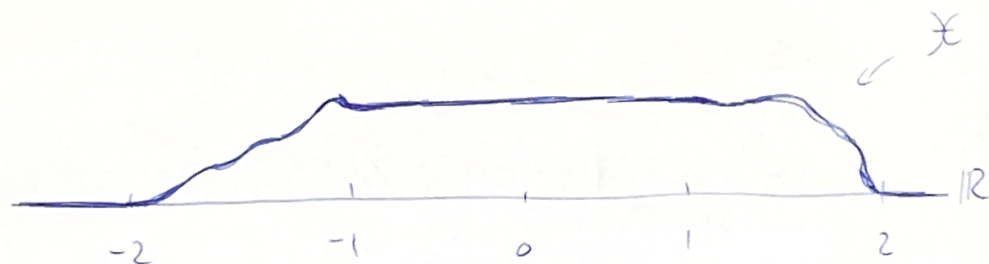
But then $\pi_h(x_0) = \psi(\pi_c(x_0))$
" " " " for all x_c , so $\phi \equiv \psi$.

(27) Note that the conditions on G in Th 3 seem a bit harsh. When are non-linearities ever bounded?

The way we will still be able to use Th 3 in bifurcation analysis is as follows: As we are often only interested in the behavior of $\dot{x} = AX + G(x) = F(x)$ near the fixed point 0, we may replace $F(x)$ by another vectorfield $\tilde{F}(x)$ that agrees with F in some open neighborhood of $0 \in \mathbb{R}^n$. To this end, let $\chi: \mathbb{R}^n \rightarrow \mathbb{R}$ be some C^∞ bump function.

That is, assume $\chi(x) = 1$ for all $x \in B_1(0) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$
and $\chi(x) = 0$ for all $x \notin B_2(0) = \{x \in \mathbb{R}^n \mid \|x\| < 2\}$

So for $n=2$ it may look like



Then we have:

Lemma 4. Given $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$ of class C^k , the map $\tilde{G}_p: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\tilde{G}_p(x) = \chi(p^{-1}x)G(x)$ agrees with G on the open set $B_p(0)$ (where $p \in \mathbb{R} > 0$)

Suppose $G(0) = 0$ & $DG(0) = 0$, then given $\varepsilon > 0$, there is a $\tilde{p}_\varepsilon > 0$ such that $\tilde{G}_p$ satisfies the two conditions 1) and 2) in Th 3, so

$$1) \sup_{x \in \mathbb{R}^n} \|D^j \tilde{G}_p(x)\| < \infty \text{ for } 0 \leq j \leq k$$

$$2) \sup_{x \in \mathbb{R}^n} \|D \tilde{G}_p(x)\| < \varepsilon$$

$$\text{If } p < \tilde{p}_\varepsilon.$$

Thus Center Manifold Reduction may be applied to the system $\dot{x} = AX + \tilde{G}_p(x)$, which agrees with $\dot{x} = AX + G(x)$ locally around 0.

(28) Proof of Lemma 4. it is clear that $\chi(p^{-1}x)G(x) = G(x)$ when $\chi(p^{-1}x) = 1$, which is (at least) when $\|p^{-1}x\| < 1$, so when $\|x\| < p$, so for $x \in B_p(0)$. also, as $\tilde{G}_p(x) = 0$ when $x \notin B_{2p}(0)$, we immediately see that

$$1) \sup_{x \in \mathbb{R}^n} \|D^j \tilde{G}_p(x)\| < \infty \text{ for } 0 \leq j \leq k, \text{ for all } p > 0.$$

Now note that $D \tilde{G}_p(x) = \chi(p^{-1}x) DG(x) + p^{-1} \chi'(p^{-1}x) G(x)$ as $G(0) = 0$ and by the Mean value theorem

$$\|G(x)\| = \|G(x) - G(0)\| \leq \sup_{s \in [0,1]} \|DG(sx)\| \|x\| \leq \|x\| \sup_{s \in [0,1]} \|DG(sx)\|$$

$$\text{So } \sup_{x \in \mathbb{R}^n} \|D \tilde{G}_p(x)\| \leq \sup_{\|x\| \leq 2p} \|D \tilde{G}_p(x)\|$$

$$\leq \sup_{\|x\| \leq 2p} \left(\|\chi(p^{-1}x)\| \|DG(x)\| + p^{-1} \|\chi'(p^{-1}x)\| \|G(x)\| \right)$$

$$\leq \sup_{\|x\| \leq 2p} (C_1 \|DG(x)\| + p^{-1} C_2 \|G(x)\|)$$

where

$$C_1 := \sup_{x \in \mathbb{R}^n} \chi(x)$$

$$C_2 := \sup_{x \in \mathbb{R}^n} \|D\chi(x)\|$$

$$\leq C_1 \sup_{\|x\| \leq 2p} \|DG(x)\| + (2p)^{-1} \sup_{\|x\| \leq 2p} \|x\| \sup_{s \in [0,1]} \|DG(sx)\|$$

$$\leq C_1 \sup_{\|x\| \leq 2p} \|DG(x)\| + C_2 p^{-1} \cdot 2p \cdot \sup_{\|x\| \leq 2p} \|DG(x)\|$$

$$= (C_1 + 2C_2) \sup_{\|x\| \leq 2p} \|DG(x)\|$$

As $DG(0) = 0$ and G is C^1 (so DG is continuous), we find

$$\lim_{p \rightarrow 0} \|D \tilde{G}_p(x)\| = \lim_{p \rightarrow 0} (C_1 + 2C_2) \sup_{\|x\| \leq 2p} \|DG(x)\| = 0.$$

So given ε , for p small enough, $\|D \tilde{G}_p(x)\| < \varepsilon$.

This completes the proof.

Before we show how to use Th 3 in an example, we now look at the coefficients of Ψ in more detail.

Note, by the way, that the construction surrounding Lemma 4 depends on all kinds of choices, and that

(29) Now Suppose we have a C^k map $\Psi: W_c \rightarrow W_h$, with W_c & W_h determined by $A \in \text{Mat}(\mathbb{R}, n)$. Suppose $\Psi(0) = 0$ and that $\{x_c + \Psi(x_c) \mid x_c \in W_c\}$ is invariant under the flow of $\dot{x} = f(x) = Ax + g(x)$, ($g(0) = 0$, $Dg(0) = 0$) on $\mathbb{R}^n = W_c \oplus W_h$. Let's see what this means for the Taylor-coefficients of Ψ . As usual, denote by π_c & π_h the projections onto W_c & W_h , respectively, along $\mathbb{R}^n = W_c \oplus W_h$. We may write $\dot{x} = f(x)$ as

$$\begin{aligned}\dot{x}_c &= A_c x_c + \pi_c g(x_c, x_h) \\ \dot{x}_h &= A_h x_h + \pi_h g(x_c, x_h)\end{aligned} \quad \text{where } x_{c/h} = \pi_{c/h}(x) \text{ and}$$

$$A = \begin{pmatrix} A_c & 0 \\ 0 & A_h \end{pmatrix} \text{ w.r.t. } \mathbb{R}^n = W_c \oplus W_h.$$

What does a vector tangent to $M_c = \{x_c + \Psi(x_c) \mid x_c \in W_c\}$ look like? well, we may describe them by $\frac{d}{dt}\big|_{t=0} (x_c + t v_c) + \Psi(x_c + t v_c)$
 $= v_c + D\Psi(x_c) v_c, v_c \in W_c.$

Thus for M_c to be flow-invariant, we need

$$\pi_h f(x_c + \Psi(x_c)) = D\Psi(x_c) \pi_c f(x_c + \Psi(x_c)) \quad \text{So}$$

$$\begin{aligned}A_h x_h + \pi_h g(x_c + \Psi(x_c)) &= D\Psi(x_c) (A_c x_c + \pi_c g(x_c + \Psi(x_c))) \\ \Rightarrow A_h \Psi(x_c) + \pi_h g(x_c + \Psi(x_c)) &= D\Psi(x_c) (A_c x_c + \pi_c g(x_c + \Psi(x_c)))\end{aligned}$$

Write $\Psi(x_c) = Bx_c + \mathcal{O}(|x_c|^2)$. with B linear, then comparing linear terms alone:

$$A_h B x_c = B A_c x_c + \mathcal{O}(|x_c|^2) \Rightarrow A_h B = B A_c \text{ on } W_c.$$

Or

$$\begin{array}{ccc} W_c & \xrightarrow{A_c} & W_c \\ B \downarrow & \curvearrowright & \downarrow B \\ W_h & \xrightarrow{A_h} & W_h \end{array}$$

as diagrams.

We claim that necessarily $B = 0$:

Lemma: Suppose $A_1: \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_1}$, $A_2: \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_2}$, $B: \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$ are linear such that A_1 & A_2 have no eigenvalues in common and suppose $A_2 B = B A_1$, then $B = 0$.

Proof

Expand A_1, A_2, B to $\mathbb{C}$ -linear maps on $\mathbb{C}^{n_1}, \mathbb{C}^{n_2}$.

Denote by $\mathcal{L}_{\mathbb{C}}(n_1, n_2)$ the space of $\mathbb{C}$ -linear maps from $\mathbb{C}^{n_1}$ to $\mathbb{C}^{n_2}$, (so $B \in \mathcal{L}_{\mathbb{C}}(n_1, n_2)$)

(30) Consider the $\mathbb{C}$ -linear map $L: \mathcal{L}_{\mathbb{C}}(n_1, n_2) \rightarrow$ given by $LX = A_2 X - X A_1$, so $B \in \ker(L)$

Claim 1 If $v_1 \dots v_{n_1}$ is a basis for $\mathbb{C}^{n_1}$ (over $\mathbb{C}$), and $w_1 \dots w_{n_2}$ is a basis for $\mathbb{C}^{n_2}$ (over $\mathbb{C}$) then a basis for $\mathcal{L}_{\mathbb{C}}(n_1, n_2)$ is given by ~~$w_i v_j^*$~~

$$\mathbb{C}^{n_2} \otimes \mathbb{C}^{*n_1}$$

$$\{w_i v_j^* \mid 0 \leq i \leq n_2, 0 \leq j \leq n_1\}$$

where $v_1^* \dots v_{n_1}^*$ is the dual basis to $v_1 \dots v_{n_1}$,

$$\text{i.e. } v_i^*(v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} = \delta_{ij}$$

Proof. Note that $\dim \mathcal{L}_{\mathbb{C}}(n_1, n_2) = n_1 n_2$. If

$$\sum_{j=1}^{n_1} \sum_{i=1}^{n_2} a_{ij} w_i v_j^* = 0 \text{ for some } a_{ij} \in \mathbb{C}, \text{ then given } 0 < k \leq n_1,$$

$$0 = \sum_{j=1}^{n_1} \sum_{i=1}^{n_2} a_{ij} w_i v_j^*(v_k) = \sum_{j=1}^{n_1} \sum_{i=1}^{n_2} a_{ij} w_i \delta_{jk} = \sum_{i=1}^{n_2} a_{ik} w_i$$

So (as the w_i are a basis) $a_{1k} = a_{2k} = \dots = a_{n_2k} = 0$.

This holds for all k , so the $w_i v_j^*$ are linearly independent. Counting dimensions, we see that we are good.

Now suppose $v_1 \dots v_{n_1}$ is a basis in which A_1 is upper-triangular, and $w_1 \dots w_{n_2}$ is a basis in which A_2 is ~~upper~~ lower-triangular, i.e.

$$A_1 v_i = d_i v_i + \text{"terms in } v_1, v_2, \dots, v_{i-1} \text{"}$$

$$A_2 w_j = \tilde{d}_j w_j + \text{"terms in } w_{j+1}, \dots, w_{n_2} \text{"}$$

Note that the d_i are the eigenvalues of A_1 and the $\tilde{d}_j$ are the eigenvalues of A_2 .

(So $d_i \neq \tilde{d}_j$ for all i, j , by assumption)

Order the basis $w_i v_j^*$ such that $w_i v_j^* < w_{l+k} v_k^*$ if $i+j < l+k$.

$$\text{e.g. } w_1 v_1^* < w_1 v_2^* < w_2 v_1^* < w_2 v_2^* < w_1 v_3^* < w_3 v_1^* < \dots$$

Claim: L is ~~upper~~ lower triangular w.r.t this ordered basis,

the diagonal entries are $\tilde{d}_j - d_i \neq 0$.

③ Proof: Recall $A_1 v_i = d_i v_i + \text{"terms lower in } i \text{"}$

$$\text{So } v_i^*(A_1 v_i) = v_i^*(d_i v_i + \text{"terms lower in } i \text{"}) = d_i = (v_i^* \circ A_1)(v_i)$$

$$\text{and } v_i^*(A_1 v_k) = v_i^*(d_k v_k + \text{"terms lower in } i \text{"}) = 0 = (v_i^* \circ A_1)(v_k) \\ \text{if } k < i.$$

$$\text{So } v_i^* \circ A_1 = d_i v_i^* + \text{"terms in } v_{i+1}^*, \dots, v_{n_1}^* \text{"}$$

$$\begin{aligned} \text{Thus } \mathcal{L} w_j v_i^* &= A_2 w_j v_i^* - w_j (v_i^* \circ A_1) \\ &= (\tilde{d}_j w_j + \text{"terms in } w_{j+1}, \dots, w_{n_2} \text{"}) v_i^* \\ &\quad - w_j (d_i v_i^* + \text{"terms in } v_{i+1}^*, \dots, v_{n_1}^* \text{"}) \\ &= (\tilde{d}_j - d_i) w_j v_i^* + \text{"higher order terms"} \end{aligned}$$

Since $\tilde{d}_j - d_i \neq 0$ for all $1 \leq i \leq n_1, 1 \leq j \leq n_2$, and since $\mathcal{L}$ is lower-triangular, we see that $\mathcal{L}: \mathcal{L}_{\mathbb{C}}(n_1, n_2)$ is injective, thus bijective. But, $\mathcal{L}(B) = 0$. So $B = 0$. $\square$

Note that we only used that A_c & A_2 (so A_c & A_h) have no eigenvalues in common!

Back to $\Psi: w_c \rightarrow w_h$, apparently

$$\Psi(x_c) = \underbrace{\Psi^{(2)}(x_c)}_{\text{2nd order}} + \underbrace{\Psi^{(3)}(x_c)}_{\text{3rd order}} + \dots \quad \begin{array}{l} \text{(No constant or linear terms} \\ \text{around } 0; \Psi(0)=0, D\Psi(0)=0) \end{array}$$

Back to our original equation

$$A_h \Psi(x_c) + \pi_h g(x_c, \Psi(x_c)) = D\Psi(x_c)(A_c x_c + \pi_c g(x_c + \Psi(x_c)))$$

$$\text{Write } \pi_h g(x_c, x_h) = g^{(2,0)}(x_c) + g^{(1,1)}(x_c, x_h) + g^{(0,2)}(x_h) + \mathcal{O}(|x_c, x_h|^3)$$

$$\text{Then } \pi_h g(x_c, \Psi(x_c)) = g^{(2,0)}(x_c) + \mathcal{O}(|x_c|^3)$$

$$\begin{aligned} \text{and } \pi_c g(x_c, \Psi(x_c)) &= \mathcal{O}(|x_c|^2), \\ D\Psi(x_c) &= \mathcal{O}(|x_c|). \end{aligned} \quad \left. \vphantom{\begin{aligned} \pi_c g(x_c, \Psi(x_c)) \\ D\Psi(x_c) \end{aligned}} \right\} D\Psi(x_c)(\pi_c g(x_c + \Psi(x_c))) = \mathcal{O}(|x_c|^3).$$

Thus for 2nd order:

$$A_h \Psi^{(2)}(x_c) + g^{(2,0)}(x_c) = D\Psi^{(2)}(x_c) A_c x_c + \mathcal{O}(|x_c|^3)$$

32 So to solve:

$$A_h \Psi^{(2)}(x_c) - D\Psi^{(2)}(x_c) A_c x_c = -g^{(2,0)}(x_c).$$

It turns out this has a unique solution $\Psi^{(2)}(x_c)$.

More generally: If $\Psi^{(2)}(x_c) \dots \Psi^{(l)}(x_c)$ are known, (as are

$\pi_{c,h} g(x_c, x_h)$) then the order $(l+1)$ terms of

$\pi_{c,h} g(x_c, \Psi(x_c))$ are known, as are those of $D\Psi(x_c)(\pi_{c,h} g(x_c, \Psi(x_c)))$

So we get the equation

$$\star \quad A_h \Psi^{(l+1)}(x_c) - D\Psi^{(l+1)}(x_c) A_c x_c = \text{"something known"}$$

Again, this has a unique solution $\Psi^{(l+1)}(x_c)$.

Thus, the Taylor-expansion of Ψ at 0 can be solved for (and is therefore unique!) up to any order that the regularity of Ψ permits! (i.e. if Ψ is C^k but not C^{k+1} then the $k+1$ order term makes little sense)

• For the uniqueness claim on eq. $\star$, see e.g.

[H.K. Wimmer, The equation $(g(x))x \circ x - b g(x) = h(x)$,
J. Math. Anal. Appl. 67 (1979), 198-204]

- Note that we only assumed a map $\Psi: W_c \rightarrow W_h$ exists (maybe only locally defined around 0) such that $\Psi(0)=0$ and the graph $M_c := \{x_c + \Psi(x_c) \mid x_c \in \overset{\circ}{U}\}$ is invariant under the flow of $\bar{x} = f(x) = A x + g(x)$. $\overset{\circ}{W}_c$

(We concluded that $D\Psi(0)=0$ and that all higher-order Taylor coefficients at 0, if they exist, are fixed by g)

We call a graph M_c of such a Ψ a Center-Manifold of the system $\bar{x} = f(x)$.

As the Taylor-coefficients around 0 are fixed, it is tempting to think center-manifolds themselves are (locally) unique. However, this is false, as

the following example shows:

Here the center manifolds will even be globally defined and C^∞ .

(33)

Consider the ODE on $\mathbb{R}^2$, given by

$$\dot{x} = -x^3$$

$$\dot{y} = -y$$

around $(0,0)$, $W_c = \{(x,y) \mid y=0\} = \{(x,0)\}$

$$W_h = \{(x,y) \mid x=0\} = \{(0,y)\}$$

Fix any $\alpha, \beta \in \mathbb{R}$, and define

$$\Psi_{\alpha,\beta}(x) = \begin{cases} \alpha e^{-\frac{1}{2x^2}} & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ \beta e^{-\frac{1}{2x^2}} & \text{for } x > 0 \end{cases} \quad (1)$$

We claim that $M_c^{\alpha,\beta} = \{(x, \Psi_{\alpha,\beta}(x)) \mid x \in \mathbb{R}\}$ is invariant for the givenODE. To see why, note that it suffices to check that the flow is tangential to (1). This is clear for $(0,0)$.

on $(x, \Psi_{\alpha,\beta}(x))$ with $x > 0$: $T_{(x, \Psi_{\alpha,\beta}(x))} M_c^{\alpha,\beta} = \{(v, D\Psi_{\alpha,\beta}(x)v) \mid v \in \mathbb{R}\}$

$$= \left\{ \left(v, \frac{\beta}{x^3} e^{-\frac{1}{2x^2}} v \right) \mid v \in \mathbb{R} \right\} \quad \text{To check: } (-x^3, -\beta e^{-\frac{1}{2x^2}}) \in T_{(x, \Psi_{\alpha,\beta}(x))} M_c^{\alpha,\beta}$$

But indeed if $v = -x^3$ then $\frac{\beta}{x^3} e^{-\frac{1}{2x^2}} v = \frac{\beta}{x^3} e^{-\frac{1}{2x^2}} (-x^3) = -\beta e^{-\frac{1}{2x^2}} \quad \checkmark$

So every choice of $\alpha, \beta \in \mathbb{R}$ defines a C^∞ center manifold.Note that these are all distinct in any neighborhood of $(0,0) \in \mathbb{R}^2$

Here another way of seeing what's going on:

Solving the ODE explicitly: $y(t) = y(0)e^{-t}$, (1)

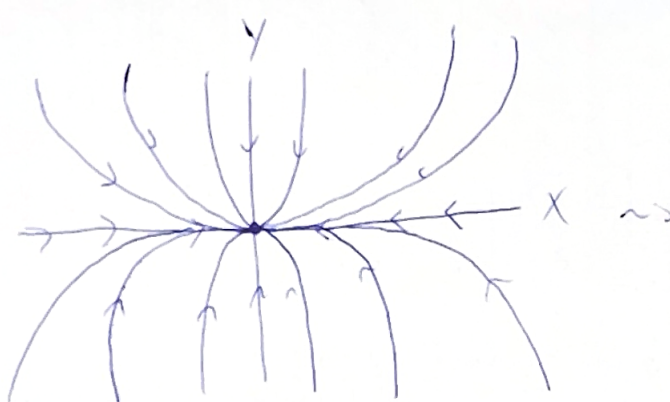
$$\frac{dx}{dt} = -x^3 \Rightarrow dt = -\frac{1}{x^3} dx \Rightarrow \int dt = \int -\frac{1}{x^3} dx$$

$$t = \frac{1}{2x^2} + c \quad (2)$$

$$\left(\text{So } 2x^2 = \frac{1}{t-c}, \quad x = \pm \frac{1}{\sqrt{2(t-c)}} = \text{sgn}(x(0)) \frac{1}{\sqrt{2(t+X(0)^2)}} \right) \quad (3)$$

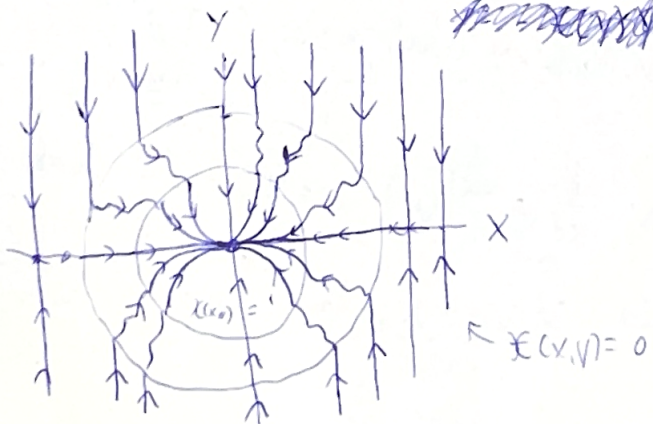
already from (1) & (2).

$$\begin{aligned} y(t) &= y(0)e^{-t} = y(0)e^{-\frac{1}{2x^2}} e^{-c} \\ &= \gamma e^{-\frac{1}{2x^2}} \quad \text{where } \gamma = y(0)e^{-c} \end{aligned}$$



So we are just bully free to make the Center-manifold out of any orbit for positive x , any orbit for negative x , and the orbit $(0,0)$. Which one would we have bound locally, using our techniques? Well, if $\chi: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a bump function then we would consider the system

$$\begin{aligned}\dot{x} &= -\chi(x)y^3 \\ \dot{y} &= -y\end{aligned}$$



Note that for big enough $\|(x,y)\|$, the ODE becomes

$$\begin{aligned}\dot{x} &= 0 \\ \dot{y} &= -y\end{aligned}$$

So to have unbounded W^h -behavior (y -behavior) (as well as an invariant graph over x) we need $\Psi(x)=0$ at least eventually. But of course $\Psi(x)=0$ everywhere works, so that is the (necessarily unique!) option we'd find: $M_c = W_c$

Alright, so we can put our finger on a (locally defined) invariant manifold containing all bounded solutions (locally). How does this help us, say in a bifurcation analysis? Well, working on a manifold is best done using a chart, luckily we have a global one.

35) Observation: a chart for M_c is given

by $w_c \ni x_c \mapsto (x_c, \psi(x_c)) \in M_c$

with inverse $M_c \ni (X_c, V_c) \mapsto X_c = \pi_c(X)$
 $\quad\quad\quad \downarrow$
 $\quad\quad\quad X \in W_c$

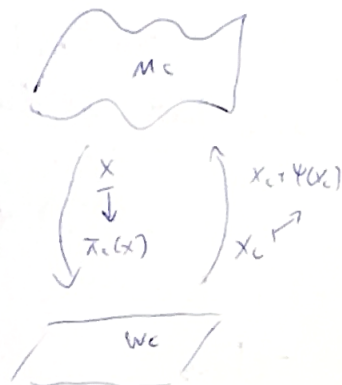


So we may pullback the vectorfield $F|_{M_c}$ to W_c .
We get the vectorfield R on W_c given by

$$R(x_c) = \pi_c F(x_c + \psi(x_c))$$

I will usually refer to R as the reduced vectorfield of the system $\dot{x} = F(x)$.

Note that $\dot{x}_c = R(x_c)$ is an ODE on W_c that is conjugate to the restriction of $\dot{x} = F(x)$ to M_c .



Writing $F(x) = F(x_c + x_h) = \begin{pmatrix} A_c x_c + g_c(x_c + x_h) \\ A_h x_h + g_h(x_c + x_h) \end{pmatrix}$

We have $R(x_c) = A_c x_c + g_c(x_c + \psi(x_c))$

Thus $R(0) = 0$, $DR(0) = A_c$, Moreover, if

$$g_c(x_c + x_h) = g_c^{(2,0)}(x_c) + g_c^{(1,1)}(x_c, x_h) + g_c^{(0,2)}(x_h) + \mathcal{O}(\|x_c + x_h\|^3)$$

Then, since $\Psi(x_i) = O(|x_i|^3)$, we get

$$R(x_c) = \underbrace{A_c x_c}_{1^{st} \text{ order}} + \underbrace{g_c^{(7,0)}(x_c)}_{2^{nd} \text{ order}} + \mathcal{O}(|x_c|^3)$$

Of course, higher order terms will depend on Ψ , in general.

Example: [From Scholarpedia.org] Consider the ODE on $\mathbb{R}^2$

given by

given by
$$\begin{cases} \dot{x} = ax^3 + xy - x^2y^2 \\ \dot{y} = -y + bx^2 + x^2y \end{cases} = F_{a,b}(x,y) \quad , (So F_{a,b}(0,0) = 0)$$

for some $a, b \in \mathbb{R}$, assume $a+b \neq 0$ (this will be clear later). Determine those a, b for which $(0,0) \in \mathbb{R}^2$ is stable

Note that $DF_{a,b}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ So the linearity gives us no information. So we will use Center Manifold Reduction

Note that $W_c = \{(x,0) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$

$W_h = \{(0,y) \mid y \in \mathbb{R}\} \subseteq \mathbb{R}^2$

We first need to calculate some Taylor expansion terms of Ψ :

Let us write $\Psi(x) = lx + cx^2 + dx^3 + \mathcal{O}(|x|^4)$. As before, we

want $\pi_h F_{a,b}(x_c + \Psi(x_c)) = D\Psi(x_c) \pi_c F_{a,b}(x_c + \Psi(x_c))$

(note that we already know $l=0$; this is just a sanity-check.)

$$\text{So } -\Psi(x) + bx^2 + x^2\Psi(x) = D\Psi(x) \left(ax^3 + x\Psi(x) - x\Psi(x)^2 \right)$$

(where $x=x_c$)

Thus

$$-lx - cx^2 - dx^3 + bx^2 + x^2(lx) + \mathcal{O}(|x|^4)$$

$$= (\cancel{l} + 2cx + 3dx^2) \left(ax^3 + lx^2 + cx^3 - x(l^2x^2) \right) + \mathcal{O}(|x|^4)$$

1st order: $-lx = 0 \Rightarrow l=0$ (as expected!)

2nd order $(b-c)x^2 = 0 \Rightarrow b=c$

3rd order $-dx^3 = 0 \Rightarrow d=0$

So $\Psi(x) = bx^2 + \mathcal{O}(|x|^4)$.

Now recall that

$$R(x) = \pi_c F_{a,b}(x + \Psi(x))$$

$$= ax^3 + x\Psi(x) - x\Psi(x)^2 + \mathcal{O}(|x|^4)$$

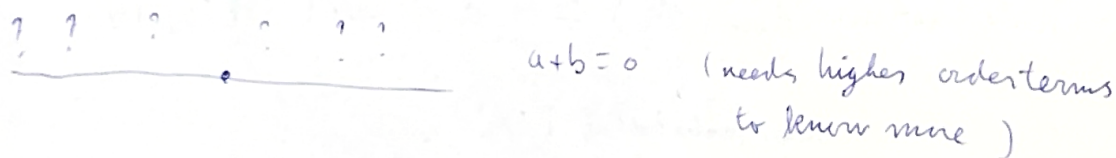
$$= ax^3 + bx^3 + \mathcal{O}(|x|^4)$$

$$= (a+b)x^3 + \mathcal{O}(|x|^4)$$

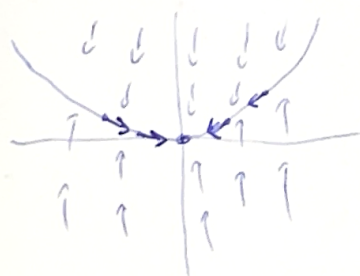


(This is why we assumed $a+b \neq 0$)

(37) We conclude that the origin is unstable if $a+b > 0$, as on the center manifold, the origin is not stable:



In fact, we will later see that we may conclude that the origin is stable for $a+b < 0$ due to the



hyperbolic eigenvalues (just -1 in this case) being stable (having negative real parts)

It should become clear that center-manifold reduction is a very useful technique, but so far we haven't done anything with bifurcations yet. So now, assume we have a family of vectorfields again: $F_\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\lambda \in \mathcal{I} \subseteq \mathbb{R}^d$

We could get centermanifolds for all systems $\dot{x} = F_\lambda(x)$ individually, but there are two problems:

- 1) How do we guarantee some kind of regularity of the resulting family of center manifolds M_c^λ , $\lambda \in \mathcal{I}$?
- 2) More importantly, $\dim(M_c^\lambda) = \dim(W_c(DF_\lambda(0)))$ may jump (it will, generically, in a bif. prob.)

(38) Instead, we look at the augmented system on $\mathbb{R}^{n+d}$

$$\begin{aligned} \dot{x} &= F_x(x) \\ \dot{\lambda} &= 0 \end{aligned} \quad \text{So } \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \tilde{F}(x; \lambda) \text{ where}$$

$$\tilde{F}: \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}, \quad \tilde{F}(x; \lambda) = (F_x(x), 0)$$

$$\text{Note that now } \dim(M_c) = \dim(W_c(D_{(x,\lambda)} \tilde{F}(0;0)))$$

center manifold at $\tilde{F}(x; \lambda)$ $n \times d$

$$\text{where } D_{(x,\lambda)} \tilde{F}(0;0) = \begin{pmatrix} D_x F_x(0;0) & D_\lambda F_x(0;0) \\ 0 & 0 \end{pmatrix}$$

So

$$\dim(W_c(D_{(x,\lambda)} \tilde{F}(0;0))) = \dim(W_c(D_x F_x(0;0))) + d$$

Note! It is tempting to think

$$W_c(D_{(x,\lambda)} \tilde{F}(0;0)) = W_c(D_x F_x(0;0) \oplus \mathbb{R}^d)$$

But this is not always the case! $\{(0; \lambda)\} \in \mathbb{R}^{n+d}$

Consider e.g.

$$F_x(x, y) = \begin{pmatrix} a\lambda + \mathcal{O}(|(x, y; \lambda)|^2) \\ -y + b\lambda + \mathcal{O}(|(x, y; \lambda)|^2) \end{pmatrix}$$

$$\text{then } D_{(x,y)} F_0(0;0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{So } \begin{aligned} W_c &= \{(x, 0) \mid x \in \mathbb{R}\} \\ W_h &= \{(0, y) \mid y \in \mathbb{R}\} \end{aligned}$$

$$\text{and } D_{(x,y,\lambda)} F_x(0;0;0) = \begin{pmatrix} 0 & 0 & a \\ 0 & -1 & b \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{then } W_h = \{(0, y; 0) \mid y \in \mathbb{R}\} \quad \text{But}$$

$$W_c = \text{span} \left\langle (1, 0; 0), \left(0, \frac{1}{b}; 1\right) \right\rangle \quad \text{if } b \neq 0.$$

↑ "a W_h -component"

③ Usually, we need an identification

$$W_c(D_{(x,\lambda)} \tilde{F}(0,0;0)) \cong W_c(D_x F_0(0)) \times \mathbb{R}^d$$

But then we get a family of reduced vectorfields:

$$R_\lambda : W_c(D_x F_0(0)) \hookrightarrow$$

—

③⑨ Usually, we need an identification

$$W_c(D_{(x,x)}\tilde{F}(0,0;0)) \cong W_c(D_x F_0(0)) \times \mathbb{R}^d$$

But then we get a family of reduced vectorfields:

$$R_\lambda : W_c(D_x F_0(0)) \hookrightarrow$$

Let's explore this a bit more. We write $A = D_x F(0,0)$

$$\tilde{A} = D_{(x,x)}\tilde{F}(0,0) \quad \text{and} \quad b = D_x F(0,0), \text{ so that}$$

$$\tilde{A} = \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \begin{matrix} \int \mathbb{R}^n \\ \int \mathbb{R}^d \end{matrix} \quad \text{as } \mathbb{R}^n \text{ sits in } \mathbb{R}^n \times \mathbb{R}^d$$

$\mathbb{R}^n$ as a subspace of $\mathbb{R}^n \times \mathbb{R}^d$.

Write W_c & W_h for the center- & hyperbolic subspaces induced by A , $\mathbb{R}^n = W_c \oplus W_h$ and

$$\tilde{W}_c \text{ & } \tilde{W}_h \text{ for those induced by } \tilde{A}, \text{ so}$$

$$\mathbb{R}^n \oplus \mathbb{R}^d = \tilde{W}_c \oplus \tilde{W}_h$$

by identifying $\mathbb{R}^n$ as " $(\mathbb{R}^n, 0)$ " in $\mathbb{R}^n \oplus \mathbb{R}^d$, we may see

W_c, W_h as subspaces of $\mathbb{R}^n \oplus \mathbb{R}^d$ too.

Then we have:

Lem: in the above setting:

$$\textcircled{1} W_h = \tilde{W}_h$$

$$\textcircled{2} W_c \subseteq \tilde{W}_c \text{ and there is a linear map}$$

$$K: \mathbb{R}^d \rightarrow W_h \text{ s.t.}$$

$$\text{Graph}(K) = \{(K(\lambda), \lambda) \mid \lambda \in \mathbb{R}^d\} \subseteq \tilde{W}_c$$

$$\text{and in fact } \tilde{W}_c = W_c \oplus \text{Graph}(K)$$

(40)

Proof: for $v \in \mathbb{R}^n$,

$$\left(\begin{array}{c|c} A & b \\ \hline 0 & 0 \end{array} \right) \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} Av \\ 0 \end{pmatrix}$$

Hence $W_c \subseteq \tilde{W}_c$, $W_h \subseteq \tilde{W}_h$. Clearly $\dim(W_h) = \dim(\tilde{W}_h)$ and $\dim(W_c) + d = \dim(\tilde{W}_c)$

So $W_h = \tilde{W}_h$. Now, write $A = \begin{pmatrix} A_{cc} & 0 \\ 0 & A_{hh} \end{pmatrix}$ w.v.t $\mathbb{R}^n = W_c \oplus W_h$

then $A_h: W_h \rightarrow W_h$ is invertible. given $\lambda \in \mathbb{R}^d$, set

$$K(\lambda) = -A_h^{-1} \pi_h b \lambda \in W_h, \text{ so } K = -A_h^{-1} \circ \pi_h \circ b$$

(Note: $b: \mathbb{R}^d \rightarrow \mathbb{R}^n$) then we have

$$\begin{aligned} \tilde{A} \begin{pmatrix} K(\lambda) \\ \lambda \end{pmatrix} &= \left(\begin{array}{c|c} A & b \\ \hline 0 & 0 \end{array} \right) \begin{pmatrix} K(\lambda) \\ \lambda \end{pmatrix} = \begin{pmatrix} AK(\lambda) + b\lambda \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -AA_h^{-1} \pi_h b \lambda + b\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} -A_h A_h^{-1} \pi_h b \lambda + b\lambda \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} b\lambda - \pi_h b \lambda \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_c b \lambda \\ 0 \end{pmatrix} \in W_c = \tilde{W}_c \end{aligned}$$

Now suppose $x \in \mathbb{R}^{n+d}$ satisfies $\tilde{A}x \in \tilde{W}_c$, then $x \in \tilde{W}_c$

For, write $x = \tilde{\pi}_h x + \tilde{\pi}_c x$, then $\tilde{A}x = \tilde{A}_h \tilde{\pi}_h x + \tilde{A}_c \tilde{\pi}_c x \in \tilde{W}_c$

$\tilde{\pi}_h|_c$ w.v.t: $\mathbb{R}^{n+d} = \tilde{W}_c \oplus \tilde{W}_h$ so $\tilde{A}_h \tilde{\pi}_h x = 0$. But $\tilde{A}_h$ is invertible, so $\tilde{\pi}_h x = 0$

So $x \in \tilde{W}_c$.

So from before, $\left\{ \begin{pmatrix} K(\lambda) \\ \lambda \end{pmatrix} \mid \lambda \in \mathbb{R}^d \right\} = \text{Graph}(K) \subseteq \tilde{W}_c$

By dimensions, (and transversality, i.e. $v \in W_c \cap \text{Graph}(K) \Rightarrow v = 0$) we find $\tilde{W}_c = W_c \oplus \text{Graph}(K)$.

Cor: Define $\pi_c|_h$ w.v.t $\mathbb{R}^n = W_c \oplus W_h$, and $\tilde{\pi}_c|_h$ w.v.t $\mathbb{R}^{n+d} = \tilde{W}_c \oplus \tilde{W}_h$, then

$$\tilde{\pi}_c(x_c + x_h; \lambda) = (x_c + K(\lambda); \lambda) \quad x_c \in W_c, x_h \in W_h, \lambda \in \mathbb{R}^d$$

$$\begin{aligned} \text{So } \tilde{\pi}_c(x; \lambda) &= (\pi_c(x) + K(\lambda); \lambda) \\ \tilde{\pi}_h(x; \lambda) &= (\pi_h(x) - K(\lambda); 0) \end{aligned}$$