(3) But we also have $B_a x_c \in W_c^{t(a)}$, so $\underbrace{B_a x_c}_{\in W_c^{t(a)}} + \underbrace{\Psi_{t(a)}(B_a x_c)}_{\in W_h^{t(a)}}$

is an element of $M_c^{t(a)}$.

So, $M_c^{t(a)}$, which is a graph over $W_c^{t(a)}$, contains both

$$\underbrace{B_a x_c}_{\in W_c^{t(a)}} + \underbrace{B_a \Psi_{s(a)}(x_c)}_{\in W_h^{t(a)}} \quad \text{and} \quad \underbrace{B_a x_c}_{\in W_c^{t(a)}} + \underbrace{\Psi_{t(a)}(B_a x_c)}_{\in W_h^{t(a)}}$$

So $B_a(\Psi_{s(a)}(x_c)) = \Psi_{t(a)}(B_a x_c) \quad \forall x_c \in W_c^{s(a)}$.

5)
$$\begin{aligned} R_{t(a)} B_a(x_c) &= \pi_c^{t(a)} F_{t(a)}(B_a x_c + \Psi_{t(a)}(B_a x_c)) \\ &= \pi_c^{t(a)} F_{t(a)}(B_a x_c + B_a \Psi_{s(a)}(x_c)) \\ &= \pi_c^{t(a)} F_{t(a)}(B_a(x_c + \Psi_{s(a)}(x_c))) \\ &= \pi_c^{t(a)} B_a F_{s(a)}(x_c + \Psi_{s(a)}(x_c)) \\ &= B_a \pi_c^{s(a)} F_{s(a)}(x_c + \Psi_{s(a)}(x_c)) \\ &= B_a R_{s(a)}(x_c) \end{aligned}$$

$(x_c \in W_c^{s(a)})$ $\square$

To conclude; assuming each F_i satisfies the conditions of the centre manifold theorem (i.e. "small nonlinearity, so that $\{x \mid \sup_{t \in \mathbb{R}} \|\pi_h^i \varphi_t^i(x)\| < \infty\}$ = "Graph$(\Psi_i : W_c^i \to W_h^i)$") then the reduced vectorfields R_i on W_c^i satisfy

$R_i :=$

1) $(R_i)_{i \in N}$ is a quiver equivariant system for the quiver Q and the Q-representation

$$W_c := \left((W_c^i)_{i \in N}, \left(B_a|_{W_c^{s(a)}} : W_c^{s(a)} \to W_c^{t(a)}\right)_{a \in A}\right)$$

2) We still have $R_i(0) = \pi_c^i F_i(0 + \Psi_i(0)) = \pi_c^i F_i(0) = 0$ $\forall i \in N$

3) The (derivative / Jacobian) of each R_i satisfies

$$DR_i(0) = \pi_c^i DF_i(0 + \Psi_i(0))(I_d + D\Psi_i(0))$$

(32) $= \pi_c^i DF_i(0)|_{W_c^i}$. In other words, write

$A_i := DF_i(0)$, and note that $A_i W_c^i \subseteq W_c^i$ & $A_i W_h^i \subseteq W_h^i$ (easy to see from the definition of W_c^i & W_h^i, or note that A_i commutes with A_i, trivially, so it follows from our earlier result on symmetries of A_i respecting W_c^i & W_h^i). So we may write

$$A_i = \begin{pmatrix} A_i^c & 0 \\ 0 & A_i^h \end{pmatrix} \quad \text{with respect to}$$

$V_i = W_c^i \oplus W_h^i$, where $A_i^c = A_i|_{W_c^i}$ and $A_i^h = A_i|_{W_h^i}$

So we simply have $\boxed{DR_i(0) = A_i^c}$

Varying A_i, and so A_i^c, we find

> The reduced system $R = (R_i)_{i \in N}$ is equivariant for Q with the representation W_c, and satisfies $R_i(0) = 0 \ \forall i \in N$, and with
>
> $DR_i(0)$ having a purely imaginary spectrum

In fact, by constructing the right F_i, we may prove that:

> Locally around 0, all possible reduced systems $R = (R_i)_{i \in N}$ are precisely all Q-symmetric systems (on some center subspaces W_c) satisfying $R_i(0) = 0 \ \forall i \in N$ and that $DR_i(0)$ has a purely imaginary spectrum $\forall i \in N$

If indeed a procedure exists for making F_i satisfy the conditions of the theorem of center manifold reduction, without loosing the symmetry and without changing F_i locally around 0

(33) Without proof, we also mention the corresponding result for quiver symmetric systems with parameters: I.e. we have a quiver $Q = A \overset{s}{\underset{t}{\rightrightarrows}} N$ and a representation $R = ((V_i)_{i \in N}, (B_a)_{a \in A})$. then a parameter family of symmetric systems is given by vectorfields

$$F_i : V_i \times \Lambda \to V_i \quad , \forall i \in N,$$

with $\Lambda \underset{\text{open}}{\subseteq} \mathbb{R}^k$ the parameter space, s.t.

$$F_{t(a)}(B_a(X), \lambda) = B_a F_{s(a)}(X, \lambda)$$

for all $X \in V_{s(a)}$, $\lambda \in \Lambda$, $a \in A$. so for every fixed value of $\lambda \in \Lambda$, the vectorfields

$$F_{i,\lambda} := F_i(\bullet, \lambda) \quad \text{form a sym. system.}$$

Suppose the systems

$$\begin{pmatrix} \dot{x}_i = F_i(x_i, \lambda) & , i \in N \\ \dot{\lambda} = 0 \end{pmatrix} \star$$

on $V_i \times \Lambda$ satisfy the conditions of the theorem guaranteeing a centermanifold

$$M_c^i = \{ y \in V_i \times \Lambda \mid \sup_{t \in \mathbb{R}} \| \pi_h \varphi_t^i(y) \| < \infty \} = \text{Graph}(\psi_i)$$

(and so assume $F_i(0,0) = 0$)

Denote by W_c^i the center subspace of the linear map

$D_x F_i(0,0) : V_i \to V_i$ (which is different from the centersubspace of the jacobian of $\star$, in subtle ways!)

Then the dynamics on $M_c^i \subseteq V_i \times \Lambda$ is ~~are~~ conjugate to a parameter dependent system on W_c^i,

$R_i(X_c^i, \lambda)$ s.t.:

1) for each fixed value of $\lambda \in \Lambda$, the systems $(R_{i,\lambda} := R_i(\bullet, \lambda))_{i \in N}$ form a quiver symmetric system for Q with the representation

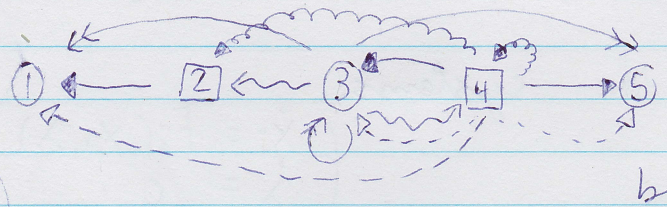
$$((W_c^i)_{i \in N}, (B_a|_{W_c^{s(a)}} : W_c^{s(a)} \to W_c^{t(a)})_{a \in A})$$

2) We have $R_i(0,0) = 0$

3) $D_x R_i(0,0)$ has a purely imaginary spectrum

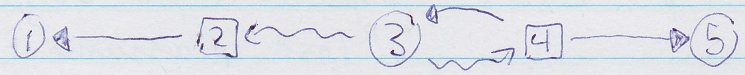
Moreover, varying over all families F_i, assuming some procedure exists for making the non-linear parts "small enough" without breaking the symmetry, and changing F_i locally around 0 we can locally get all vectorfields on W_i^i, $\forall i \in N$, satisfying the three parts above (1), 2) & 3)).

<u>Example</u>: Consider the network



, which might look convoluted, but it is just the network

(2 node types "○" and "□")



with backwards concatenated arrow types added, i.e. ○ ← □ ← ○ gives ○ ⇄ ○, etc.

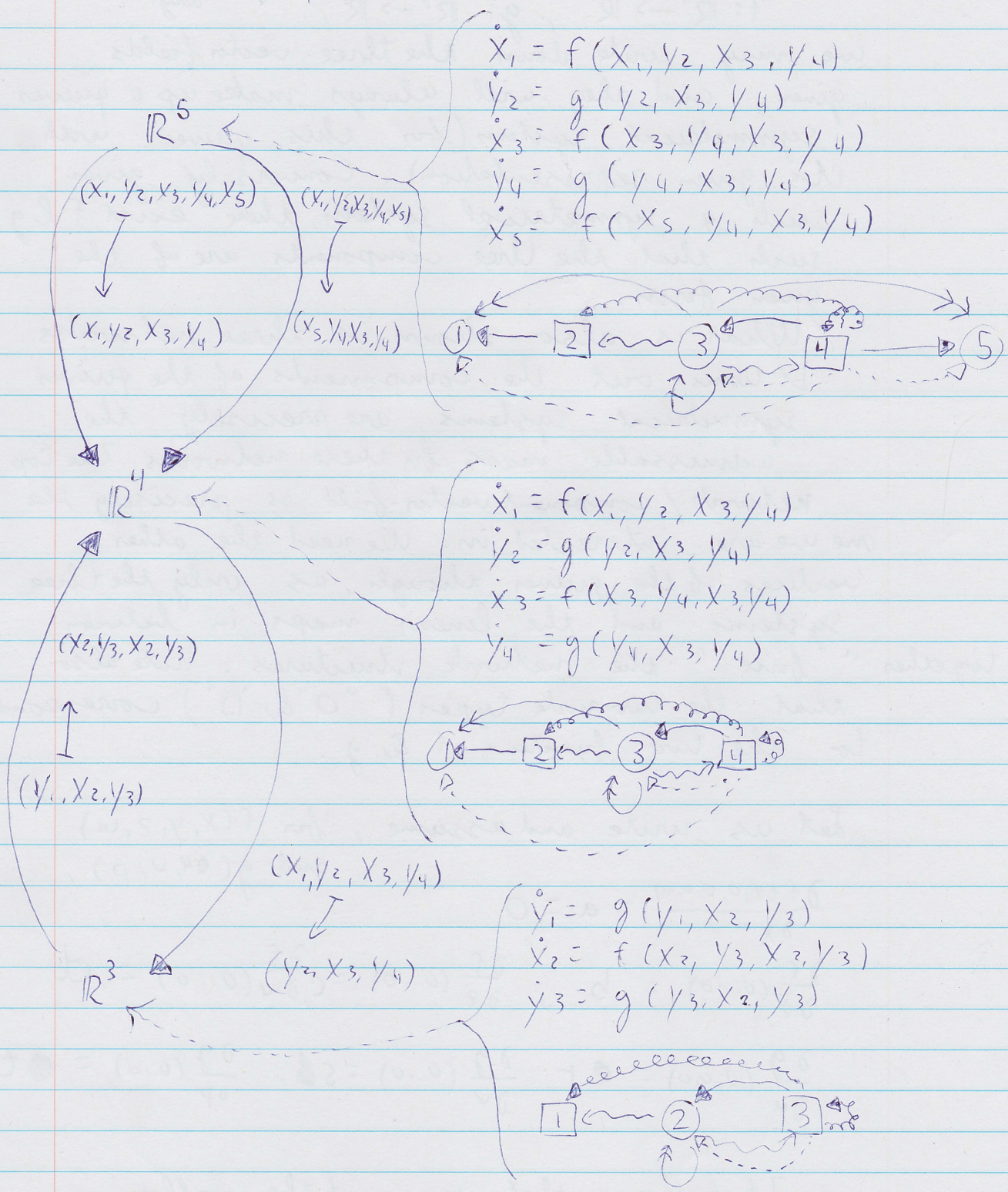
ODE's admissible for the first system are given by

$$\dot{x}_1 = f(x_1, y_2, x_3, y_4)$$
$$\dot{y}_2 = g(y_2, x_3, y_4)$$
$$\dot{x}_3 = f(x_3, y_4, x_3, y_4)$$
$$\dot{y}_4 = g(y_4, x_3, y_4)$$
$$\dot{x}_5 = f(x_5, y_4, x_3, y_4)$$

say, with $f: \mathbb{R}^4 \to \mathbb{R}$, $g: \mathbb{R}^3 \to \mathbb{R}$

where x_i corresponds to a round node "○" and y_i to a square node "□"

(35) To analyse this system, we use the following quiver representation (we will explain it a bit more below the picture)



On the left is shown a particular quiver, together with a representation of that quiver. On the right are given three vectorfields that

(36) together make up a quiver symmetrical system for this quiver with its given representation. More precisely, given any functions

$$f: \mathbb{R}^4 \to \mathbb{R}, \quad g: \mathbb{R}^3 \to \mathbb{R},$$

we may write down the three vector fields given, and they will always make up a quiver symmetrical system (for this quiver with its given representation). Conversely, given such a symmetrical system, there exist f & g such that the three components are of the given form.

What is also shown is three networks. It turns out the components of the quiver symmetrical systems are precisely the admissable maps for these networks. The top network / component-vector-field is precisely the one we are interested in. We need the other vertices of the quiver though, as only the three systems and the linear maps in between together "force" the network structures. Note also that the two node types ("O" & "□") correspond to the two functions f & g.

Let us write and assume, for $f(x, y, z, w)$ and $g(u, v, p)$,

$$\frac{\partial f(x, 0, 0, 0)}{\partial x} = a = 0$$

$$\frac{\partial f}{\partial y}(0 \ldots 0) = b, \quad \frac{\partial f}{\partial z}(0 \ldots 0) = c, \quad \frac{\partial f}{\partial w}(0 \ldots 0) = d$$

$$\frac{\partial g}{\partial u}(0, 0, 0) = r \quad \frac{\partial g}{\partial v}(0 \ldots 0) = s, \quad \frac{\partial g}{\partial p}(0 \ldots 0) = t$$

It follows that we get the following three Jacobians and center subspaces:

(37)

$$\begin{pmatrix} \overset{0}{\cancel{a}} & b & c & d & 0 \\ 0 & r & s & t & 0 \\ 0 & 0 & \overset{0}{\cancel{a+c}} & b+d & 0 \\ 0 & 0 & s & v+t & 0 \\ 0 & 0 & c & b+d & \overset{0}{\cancel{d}} \end{pmatrix} \rightsquigarrow W_c^{\text{"}5\text{"}} \qquad \boxed{2\text{ dim}}$$

$$W_c^{\text{"}5\text{"}} = \{(x_1, y_2, x_3, y_4, x_5) \mid y_2 = x_3 = y_4 = 0\} = \{(x_1, 0, 0, 0, x_5)\} \subseteq \mathbb{R}^5$$

$$(x_1,0,0,0,x_5) \mapsto (x_1,0,0,0) \qquad (x_1,0,0,0,x_5) \mapsto (x_5,0,0,0)$$

$$\begin{pmatrix} \overset{0}{\cancel{a}} & b & c & d \\ 0 & v & s & t \\ 0 & 0 & c & b+d \\ 0 & 0 & s & v+t \end{pmatrix} \rightsquigarrow W_c^{\text{"}4\text{"}} = \{(x_1, 0, 0, 0)\} \subseteq \mathbb{R}^4 \qquad \boxed{1\text{ dim}}$$

$$0 \qquad (x_1,0,0,0) \mapsto 0$$

$$\begin{pmatrix} v & s & t \\ 0 & \overset{0}{\cancel{d+c}} & b+d \\ 0 & s & v+t \end{pmatrix} \rightsquigarrow W_c^{\text{"}3\text{"}} = \text{"}0\text{"} \subseteq \mathbb{R}^3 \qquad \boxed{0\text{ dim}}$$

Shown are also the restrictions of the linear maps to the different $W_c^{\text{"}\cdot\text{"}}$. Note that these indeed maps centre subspaces into centre subspaces.
Identifying $W_c^{\text{"}5\text{"}} \cong \mathbb{R}^2$, $W_c^{\text{"}4\text{"}} \cong \mathbb{R}$, $W_c^{\text{"}3\text{"}} \cong \{0\}$, we get the following <u>subrepresentation</u>

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, bend left, "(x,y)\mapsto x"] \arrow[r, bend right, "(x,y)\mapsto y"'] & \mathbb{R} \arrow[r, bend left, "0"] & 0 \arrow[l, bend left, "0"] \end{tikzcd}$$

Let us determine all quiver symmetric systems on it, as these are our (potential) reduced vector fields.

(38) A general triple of maps looks like

$$\begin{tikzcd}
\begin{pmatrix} k(x,y) \\ l(x,y) \end{pmatrix} \arrow[r, bend left, "(x,y)\mapsto x"] \arrow[r, bend right, "(x,y)\mapsto y"'] & h(z) \arrow[r, bend left, "0"] & 0 \arrow[l, bend left, "0"]
\end{tikzcd}$$

The map $(x,y)\mapsto x$ imposes the condition:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "(x,y)\mapsto x"] \arrow[d, "(x,y)\mapsto \begin{smallmatrix} k(x,y) \\ l(x,y) \end{smallmatrix}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R} \arrow[d, "\begin{smallmatrix} z \\ \downarrow \\ h(z). \end{smallmatrix}"] \\
\mathbb{R}^2 \arrow[r, "(x,y)\mapsto x"] & \mathbb{R}
\end{tikzcd}$$

Imposing the diagram commutes gives

$$\left.\begin{array}{ll} \hookrightarrow: & k(x,y) \\ & \quad \| \\ \rightarrow: & h(x) \end{array}\right\} \begin{array}{c} k(x,y) \\ \| \\ h(x). \end{array}$$

Likewise the map $(x,y)\mapsto y$ induces

$$l(x,y) = h(y)$$

The zero map $\xrightarrow{0}$ always sends solutions to solutions.

The other zero map precisely when $h(0)=0$.

Summarising, the symmetric systems on this quiver representation are given precisely by

$$\begin{tikzcd}
\begin{pmatrix} \dot{x} = h(x) \\ \dot{y} = h(y) \end{pmatrix} \arrow[r, bend left, "(x,y)\mapsto x"] \arrow[r, bend right, "(x,y)\mapsto y"'] & \begin{pmatrix} \dot{z} = h(z) \end{pmatrix} \arrow[r, bend left, "0"] & \dot{0} \arrow[l, bend left, "0"]
\end{tikzcd}$$

for any $h:\mathbb{R}\to\mathbb{R}$ satisfying $h(0)=0$

Now, if we added a bifurcation paramater $\lambda\in\mathbb{R}$, ...

(39) ... So if we looked at the systems

$$\begin{aligned}
\dot{x}_1 &= f(x_1, y_2, x_3, y_4; \lambda) \\
\dot{y}_2 &= g(y_2, x_3, y_4; \lambda) \\
\dot{x}_3 &= f(x_3, y_4, x_3, y_4; \lambda) \quad \text{with } \frac{\partial f}{\partial x_1}(0,0,0,0;0) = 0 \\
\dot{y}_4 &= g(y_4, x_3, y_4; \lambda) \\
\dot{x}_5 &= f(x_5, y_4, x_3, y_4; \lambda)
\end{aligned}$$

then we'd get the family of reduced vectorfields

$$\dot{x} = h(x;\lambda) \qquad \dot{z} = h(z;\lambda), \qquad \text{"0"}$$
$$\dot{y} = h(y;\lambda)$$

Where $h(0;\lambda) = 0 \quad \forall \lambda \in \mathbb{R}$, because of our earlier result (i.e. in case of a bifurcation parameter, we may simply take families of the reduced vectorfields we'd get without paramaters)

We also have $h(0;0) = 0$ (but this already follows from $h(0;\lambda) = 0 \ \forall \lambda \in \mathbb{R}$) and $\begin{pmatrix} D_x h(0;0) & 0 \\ 0 & D_x h(0;0) \end{pmatrix}$ has a purely imaginary spectrum. This last claim is equivalent to $D_x h(0;0) = 0$

Hence, we may now perform a "bare hands analysis" to find all branches of steady states.

To this end, note that $h(0;\lambda) = 0$ implies we may write

$$h(x;\lambda) = x\tilde{h}(x;\lambda)$$

$$\begin{aligned}
&= x(a(\lambda) + b(\lambda)x + \mathcal{O}(|x|^2)) \\
&= x(\underbrace{\cancel{a_0}}_{0} + a_1\lambda + b_0 x + b_1 x\lambda + \mathcal{O}(|x|^2 + |\lambda|^2))
\end{aligned}$$

$0 \leftarrow$ as $D_x h(0;0) = 0$

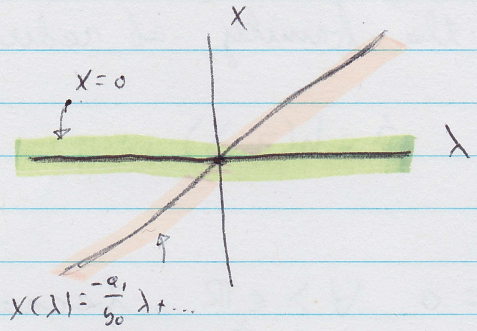
We find the branches "$x = 0$", $x(\lambda)$, and solving

$$a_1\lambda + b_0 x + \mathcal{O}(|x|^2 + |\lambda||x| + |\lambda|^2) = 0 \quad \ldots$$

(40) By the implicit function theorem, assuming $b_0 \neq 0$, we get the additional branch of steady states

$$x(\lambda) = \frac{-a_1}{b_0}\lambda + O(|\lambda|^2)$$

Hence the picture (for $\dot{x} = h(x;\lambda)$) is



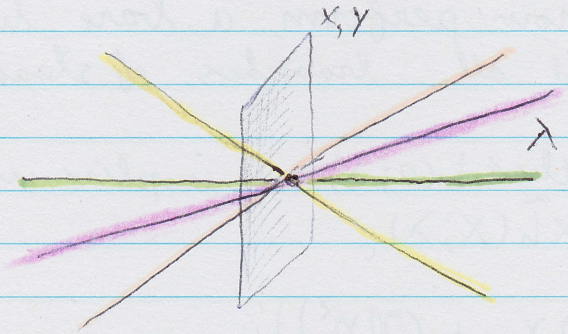
The reduced dynamics of our system of interest (on $\mathbb{R}^5$) is given by

$$\dot{x} = h(x;\lambda)$$
$$\dot{y} = h(x;\lambda)$$

So we get 4 branches:

$(x,y)(\lambda) = (0,0),\ (0, \frac{-a_1}{b_0}\lambda + \dots),\ (\frac{-a_1}{b_0}\lambda + \dots, 0),\ (\frac{-a_1}{b_0}\lambda + \dots, \frac{-a_1}{b_0}\lambda + \dots)$

So the picture is



We may even understand the whole phase portrait. Focussing on a fixed value $\lambda = \lambda_0 \neq 0$ (close enough to $\lambda = 0$), a standard forcing theorem and the fact that we know all steady state points close to $x = 0$ gives ...

(41) A connecting (heteroclinic) orbit between the two fixed points of $h(x,\lambda_0)$ (, in some direction, depending on the stability of the two points).

Let us denote the two steady state points of $h(x,\lambda_0)$ by x_0, x_1 (so $\{x_0, x_1\} = \{0, \frac{-a_1}{b_0}\lambda_0 + \dots\}$) then the picture is (assuming $x_0 < x_1$)

For $\tilde{x} \in (x_0, x_1)$, denote by $\varphi_t(\tilde{x})$ the orbit through $\tilde{x}$, so that

$$\lim_{t\to\infty} \varphi_t(\tilde{x}) = x_0, \quad \lim_{t\to-\infty} \varphi_t(\tilde{x}) = x_1 \quad \text{or}$$

$$\text{"} \qquad \text{"} = x_1, \quad \text{"} \qquad \text{"} = x_0.$$

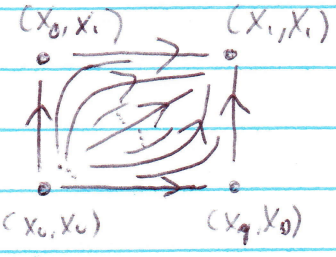
Then we find infinitely many solutions to

$$\dot{x} = h(x,\lambda_0) \quad \star$$
$$\dot{y} = h(y,\lambda_0)$$

Namely $\hat{\varphi}_t(\tilde{x},\tilde{y}) := (\varphi_t(\tilde{x}), \varphi_t(\tilde{y}))$ for all $\tilde{x}, \tilde{y} \in (x_0, x_1)$

(Though not all give different flow lines)

The picture to have is e.g.



so we get infinitly many connecting orbits between (x_0,x_0) & (x_1,x_1) as well as connections between (x_0,x_0) & (x_0,x_1) and between (x_0,x_0) and (x_1,x_0), etc. Note that we use here that the reduced system $\star$ is <u>totally decoupled</u>, which we <u>cannot</u> conclude from looking at (finitely many) Taylor-coefficients. Rather we used Quiver symmetry! (To get the exact result of decoupled dynamics)