Center manifold Reduction

by Eddie Nyhuis

Preliminaries

- Let $A \in \mathbb{R}^{n\times n}$ be a linear map from $\mathbb{R}^n$ to itself. We may also see A as a $\mathbb{C}$-linear map from $\mathbb{C}^n$ to itself (i.e. $A(v+iw) = Av + iAw$ for $v, w \in \mathbb{R}^n$)
  As such, $\mathbb{C}^n$ has a (complex) basis of eigenvectors and generalised eigenvectors of A, $\{v_1 \dots v_n\}$

Set $\widetilde{W}_H = \operatorname{span}_{\mathbb{C}}\{v_i \mid v_i$ is a (generalised) eigenvector for an eigenvalue λ of A not on the imaginary axis $i\mathbb{R} \subseteq \mathbb{C}\}$



and

$\widetilde{W}_C = \operatorname{span}_{\mathbb{C}}\{v_i \mid$ " — " of A on $i\mathbb{R} \subseteq \mathbb{C}\}$

Then easy to see $\mathbb{C}^n = \widetilde{W}_H \oplus_{\mathbb{C}} \widetilde{W}_C$

Moreover, set $W_H = \widetilde{W}_H \cap \mathbb{R}^n$, $W_C = \widetilde{W}_C \cap \mathbb{R}^n$

then $\mathbb{R}^n = W_H \oplus_{\mathbb{R}} W_C$

(Proof. [sketch] note that $v \in \widetilde{W}_H \iff \bar{v} \in \widetilde{W}_H$

$v \in \widetilde{W}_C \iff \bar{v} \in \widetilde{W}_C$

then for $x \in \mathbb{R}^n$, write $x = u + w = \bar{u} + \bar{w}$

for $u \in \widetilde{W}_H$, $w \in \widetilde{W}_c$, by uniqueness,
$u = \bar{u}$, $w = \bar{w}$)

Def: · W_H is the hyperbolic subspace of A
· W_c is the center subspace of A
· Let $\pi_c : \mathbb{R}^n = W_H \oplus W_c \to W_c$
projection onto W_c.
$\pi_H : \mathbb{R}^n = W_H \oplus W_c \to W_H$
projection onto W_H
so $\pi_c(x) = x$ for $x \in W_c$, $\pi_c(x) = 0$ for $x \in W_H$
etc., $\pi_c + \pi_H = Id_{\mathbb{R}^n}$ (with slight abuse of notation).
(We will not use $\widetilde{W}_c$ or $\widetilde{W}_H$ anymore)

Center Manifold Reduction

Consider an ODE $\dot{x} = F(x)$, $x \in \mathbb{R}^n$
with $F : \mathbb{R}^n \to \mathbb{R}^n$ of class C^k
(k-times continuously differentiable) for some
$k \geq 1$, Suppose also that $F(0) = 0$.

We may write $\dot{x} = F(x) = Ax + G(x)$
with $A = DF(0)$ (Jacobian)
and (thus) $DG(0) = 0$, $G(0) = 0$,
· with G also C^k

Let W_c, $W_H \subseteq \mathbb{R}^n$, π_c, π_H etc.
· be the corresponding spaces/maps
for A

Thm [Center manifold Reduction]

Let $\|\cdot\|$ denote the standard norm on $\mathbb{R}^n$

Given $A \in \mathbb{R}^{n\times n}$ and $k \in \mathbb{N}_{(\geq 1)}$ there exists $\varepsilon = \varepsilon(A,k) > 0$ s.t.:

If $G : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^k with $G(0) = 0$ & $DG(0) = 0$, and satisfies

1) $\sup_{x\in\mathbb{R}^n} \|D^j G(x)\| < \infty$ for $0 \leq j \leq k$ (operator norm) (bounded derivatives)

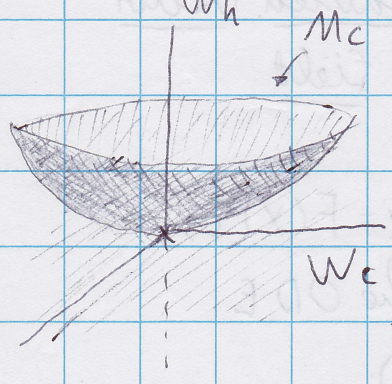
2) $\sup_{x\in\mathbb{R}^n} \|DG(x)\| < \varepsilon$ ("extra" bounded derivatives)

Then there exists a C^k map $\Psi : W_c \to W_h$ s.t.

$$M_c = \text{"Graph}(\Psi)\text{"} := \{ x_c + \Psi(x_c) \mid x_c \in W_c \} \subseteq W_c \oplus W_h = \mathbb{R}^n$$

is invariant under the flow of $\dot{x} = F(x) = Ax + G(x)$

so if $\varphi_t(x)$ satisfies $\frac{d}{dt}\varphi_t(x) = F(\varphi_t(x))$ and $x \in M_c$ then $\varphi_t(x) \in M_c \ \forall\, t \in \mathbb{R}$



Moreover, we have (under the conditions of the theorem)

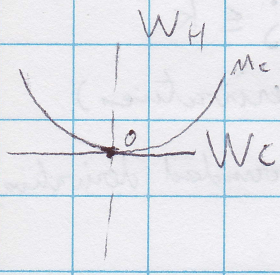
$$M_c = \left\{ x \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty \right\}$$

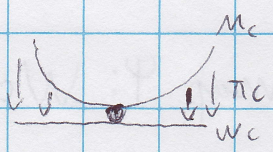
! In particular, M_c contains <u>all bounded solutions</u> to $\dot{x} = F(x)$ (but possibly more solutions, of course)

such as steady state points & periodic orbits

Finally, $\Psi(0)=0$ and $D\Psi(0)=0$, so that M_c touches 0

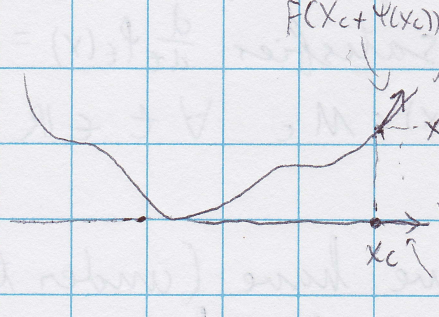
Note that M_c is a manifold of dimension $\dim(W_c)$, a global coordinate chart is given by $M_c \ni X \longmapsto \pi_c(X) \in W_c$ with inverse $W_c \ni x_c \longmapsto x_c + \Psi(x_c) \in M_c$





Correspondingly, the dynamics on M_c is conjugate to that of the push-forward on W_c of $F|_{M_c}$ by $\pi_c : M_c \to W_c$. We will denote this by $R(x_c)$, $x_c \in W_c$, so

$$R(x_c) = \pi_c F(x_c + \Psi(x_c))$$



I usually call R the Reduced vector field

"Interesting Behaviour" of $\dot{x} = F(x)$ is therefore captured in the ODE

$$\dot{x}_c = R(x_c)$$

⑤ • The main problem with the theorem on center manifold reduction is the strong conditions on the nonlinear part $G(x)$.

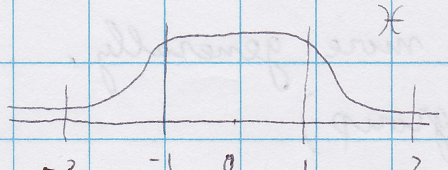
I.e. given a system $\dot{x} = F(x) = Ax + G(x)$ you don't expect $\|D^k G(x)\| < \varepsilon(A,k)$ for all $x \in \mathbb{R}^n$, nor for G and it's k-derivatives to be bounded. If you are only interested in behavior of $\dot{x} = F(x)$ around $x=0$, locally, then there is a way around this.

Let $F(x) = Ax + G(x)$ be given, $\left(\begin{array}{l} F(0) = G(0) = 0 \\ DG(0) = 0,\ A = DF(0) \end{array}\right)$

with G not necessarily satisfying the conditions of Center Manifold Reduction

We will create $\tilde{F}(x) = Ax + \tilde{G}(x)$ with $\tilde{G}$ satisfying the conditions, and $F|_U = \tilde{F}|_U$ for some open neighborhood U of $0 \in \mathbb{R}^n$

To this end, let $\chi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth bump function, i.e. e.g. $\chi(y) = \begin{cases} 0 & \text{if } \|y\| \geq 2 \\ 1 & \text{if } \|y\| \leq 1 \end{cases}$



for any $\rho \in \mathbb{R}_{>0}$, set

$$\tilde{G}_\rho(x) := \underbrace{\chi(\rho^{-1}\|x\|)}_{\text{scalar}} \cdot G(x)$$

then • $\tilde{G}_\rho(x) = G(x)$ when $\|x\| \leq \rho$ (so $F(x) = Ax + G(x) = Ax + \tilde{G}_\rho(x)$ for $\|x\| \leq \rho$)

and

• $\sup_{x \in \mathbb{R}^n} \|D^j \tilde{G}_\rho(x)\| < \infty$ for all $0 \leq j \leq k$.

- Finally $\lim_{\rho \to 0} \left(\sup_{x \in \mathbb{R}^n} \| D \tilde{G}_\rho(x) \| \right) = 0$,

so that for small enough ρ_0,
$\tilde{G}(x) := \tilde{G}_{\rho_0}(x)$ and
$\tilde{F}(x) = Ax + \tilde{G}(x)$ does the trick
(with $U = \{ x \mid \|x\| \leq \rho_0 \}$)

This is what we typically mean by Local Center manifold Reduction

(See for instance [A. Vanderbauwhede, Centre manifolds, normal forms and elementary bifurcations. Dynamics Reported 2 (1989)] for proofs and more material)

Of course problems arise when $F(x)$ has some structure that you'd like to preserve, (so at any rate in $\tilde{F}(x)$), for instance a symmetry:

- Let G be a finite group (Or more generally, a compact topological group)

- Suppose we have a left action of G on $\mathbb{R}^n$ by linear operations: $g \mapsto C_g \in \mathcal{L}in(\mathbb{R}^n, \mathbb{R}^n)$
$C_g \cdot C_h = C_{gh}$, $C_{g^{-1}} = C_g^{-1}$, $C_e = Id_n$ $\forall g, h \in G$, e the unit of G, so $\mathbb{R}^n$ is a representation space of G.

(7) (for compact, assume the action is continues)

- An ODE has a symmetry (given by $(C_g)_{g\in G}$) if the corresponding vector field commutes with the maps C_g, so if $\dot{x} = F(x)$ then $F \circ C_g = C_g \circ F$ for all $g \in G$. This means that if $\varphi_t(x)$ is a solution to the ODE, then so is $C_g \varphi_t(x)$, for any $g \in G$

- writing $F(x) = Ax + G(x)$ with $A = DF(0)$, and $G(0) = 0$, one easily shows that $A \circ C_g = C_g \circ A \quad \forall g \in G$ (Proof: differentiate $F \circ C_g = C_g \circ F$ to get $DF(C_g(0))\, DC_g(0) = DC_g(F(0))\, DF(0)$ $\Rightarrow DF(0) \cdot C_g = C_g \cdot DF(0)$)

- So it also holds that $G \circ C_g = C_g \circ G \quad \forall g \in G$
But, for $\tilde{G}(x) := \tilde{G}_{\rho_0}(x) := \chi(\rho_0^{-1}\|x\|)\, G(x)$ we need not necessarily have $\tilde{G} \circ C_g = C_g \circ \tilde{G}$.
However, as we assumed G to be finite (or compact), there exists an inner-product $\langle \cdot, \cdot \rangle_G$ (and norm $\|x\|_G := \sqrt{\langle x, x \rangle_G}$) such that $\langle C_g x, C_g x \rangle_G = \langle x, x \rangle_G$ for all $x \in \mathbb{R}^n$, $g \in G$.
Therefore $\|C_g x\|_G = \|x\|$ for all $x \in \mathbb{R}^n$, $g \in G$.
Setting $\tilde{G}(x) := \chi(\rho_0^{-1}\|x\|_G)\, G(x)$ for any $\rho_0 > 0$, we indeed find

$$\begin{aligned}\tilde{G}(C_g(x)) &= \chi(\rho_0^{-1}\|C_g x\|_G)\, G(C_g(x)) \\ &= \chi(\rho_0^{-1}\|x\|_G)\, C_g G(x) \\ &= C_g\, \chi(\rho_0^{-1}\|x\|_G)\, G(x) \\ &= C_g\, \tilde{G}(x)\end{aligned}$$

so that this works.

(8) We will see later that this symmetry can be transferred all the way to R

First another example of a structure to be preserved:

Suppose we have the ODE on $\mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(x,y) \end{aligned} \qquad 1)$$

Then the non-linear part (i.e the "G in $F(F^{-1}Ax + G(x)$") has the same structure

However, $\chi(\rho^{-1}\|(x,y)\|)$ now looks like

$$\begin{pmatrix} \chi(\rho^{-1}\|(x,y)\|)\,\hat{f}(x) \\ \chi(\rho^{-1}\|(x,y)\|)\,\hat{g}(x,y) \end{pmatrix} \quad \left(\text{if } G(x) = \begin{pmatrix} \hat{f}(x) \\ \hat{g}(x,y) \end{pmatrix} \right)$$

which is not of the required form anymore (i.e. $\dot{x}$ now depends on y as well)

However, what works here is of course to define $\tilde{G}(x)$ as

$$\tilde{G}(x) = \begin{pmatrix} \chi(\rho^{-1}|x|) \cdot \hat{f}(x) \\ \chi(\rho^{-1}\|(x,y)\|) \cdot \hat{g}(x,y) \end{pmatrix}$$

Again (in a way) the structure of 1) can be preserved all the way to $\mathbb{R}$, as we will see

(9) ==

Let's first do some examples of Centre manifold reduction:

1) [From scholarpedia.org]

Consider the ODE on $\mathbb{R}^2$ given by

$$\dot{x} = ax^3 + xy - xy^2$$
$$\dot{y} = -y + bx^2 + x^2 y \quad \text{for some } a, b \in \mathbb{R}$$

write $F_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ for the vector-field on the right

then $F_{a,b}(0) = 0$ and

$$DF_{a,b}(0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = A$$

Suppose we want to know if 0 is stable for different values of $a, b \in \mathbb{R}$. We cannot tell from $DF_{a,b}(0)$ alone.

So, (without worrying about global existence) let us see what a centre manifold would look like.

from $A = DF_{ab}(0)$ we see that

$$W^c = \{ (x, 0) \mid x \in \mathbb{R} \} \subseteq \mathbb{R}^2$$
$$W^h = \{ (0, y) \mid y \in \mathbb{R} \} \subseteq \mathbb{R}^2$$

So a local centre manifold is given by

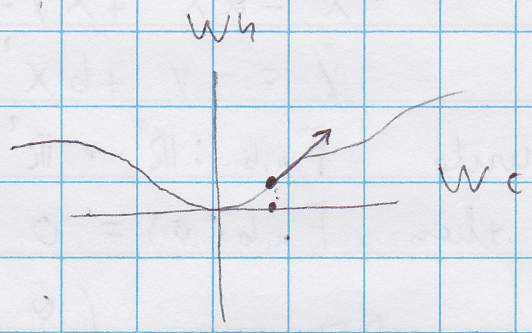
$$M^c = \{ (x, \Psi(x)) \mid x \in U \subseteq \mathbb{R} \} \subseteq \mathbb{R}^2$$

with U open in W^c

and $0 \in U$.

Also, $\Psi(0)=0$ and $D\Psi(0)=0$

Finally, recall that M_c has to be invariant under the dynamics of $\ddot{X} = F_{ab}(X)$.
This means that for any point $X=(x,\Psi(x))$ in M_c, the vectorfield $F_{ab}(X)$ has to lie in the tangentspace $T_X M_c$ of M_c in X



It can be shown that $T_X M_c = \{(v, D\Psi(x)v) \mid v \in \mathbb{R}\} \subseteq T_X \mathbb{R}^2 \cong \mathbb{R}^2$
Hence, we get the condition on Ψ

$$\pi_h\left(F_{ab}(X,\Psi(x))\right) = D\Psi(x)\,\pi_c F_{ab}(X,\Psi(x))$$

so

$$\star \quad -\Psi(x) + bx^2 + x^2\Psi(x) = D\Psi(x)\cdot(ax^3 + x\Psi(x) - x\Psi(x)^2)$$

Writing $\Psi(x) = cx^2 + dx^3 + \mathcal{O}(|x|^4)$, we get

$$-cx^2 - dx^3 + bx^2 + \mathcal{O}(|x|^4) = \mathcal{O}(|x^4|)$$

so $(b-c)x^2 - dx^3 = 0 \Rightarrow b=c,\ d=0,$

so $\Psi(x) = bx^2 + \mathcal{O}(|x|^4)$

(In fact, writing $\Psi(x) = lx + cx^2 + dx^3 + \mathcal{O}(|x|^4)$, Equation $\star$ would give us $l=0$ as well)

Recall that $R(x) = \pi_c F_{ab}(x, \Psi(x))$

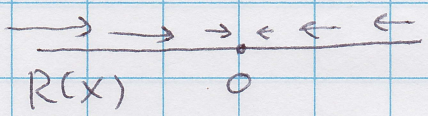
so $R(x) = ax^3 + x\Psi(x) - x(\Psi(x))^2$

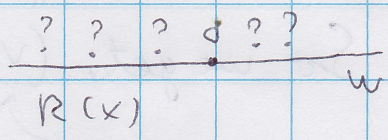
$= ax^3 + x\,bx^2 - x\,b^2x^4 + \mathcal{O}(|x|^5)$

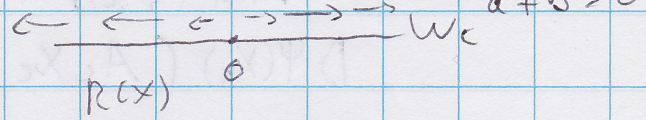
$= ax^3 + bx^3 + \mathcal{O}(|x|^5)$

$= (a+b)x^3 + \mathcal{O}(|x|^5)$

We conclude that $(0,0) \in \mathbb{R}^2$ is stable when $a+b<0$ and unstable when $a+b>0$







(when $a+b=0$ we still cannot tell, and will have to calculate $\Psi(x)$ to higher order)

As suggested by the previous example, the Taylor coefficients of Ψ are fixed, even in local centre manifolds.

To see why, write $\dot{x} = F(x) = Ax + G(x)$ as

$$\begin{pmatrix} \dot{x}_c \\ \dot{x}_h \end{pmatrix} = \begin{pmatrix} A_c & 0 \\ 0 & A_h \end{pmatrix} \begin{pmatrix} x_c \\ x_h \end{pmatrix} + \begin{pmatrix} G_c(x_c, x_h) \\ G_h(x_c, x_h) \end{pmatrix} \quad \text{with}$$

$x_c \in W_c$, $x_h \in W_h$ and $A_c = A|_{W_c} : W_c \to W_c$

$A_h = A|_{W_h} : W_h \to W_h$

(12) Here I have used the fact that A sends W_c to itself and W_h to itself, which follows from the definitions of W_c and W_h.

Moreover, $G_c := \pi_c G$ and $G_h := \pi_h G$

As in the example, we require that

$$\left(T_{x_c+\Psi(x_c)}\mathbb{R}^n \ni\right) F(x_c+\Psi(x_c)) \in T_{x_c+\Psi(x_c)} M_c$$

$$\| $$

$$\{(v_c, D\Psi(x_c)v_c) \mid v_c \in W_c\}$$

So we get: ($v_h = D\Psi(x_c)v_c$)

$$A_h\Psi(x_c) + G_h(x_c, \Psi(x_c)) = \qquad (\star\star)$$
$$D\Psi(x_c)\underbrace{\left(A_c x_c + G_c(x_c, \Psi(x_c))\right)}_{v_c}$$

Write

$$\Psi(x_c) = Lx_c + \Psi^{(2)}(x_c) + \Psi^{(3)}(x_c) + \dots$$

with $L : W_c \to W_h$ linear, $\Psi^{(k)}(x_c)$ order k-terms, then $(\star\star)$ becomes

$$A_h(Lx_c + \dots) + G_h(x_c, Lx_c + \dots) =$$
$$\left(L + D\Psi^{(2)}(x_c) + \dots\right)\left(A_c x_c + G_c(x_c, Lx_c + \dots)\right)$$

1st order $\quad A_h L = L A_c$, this actually implies $L = 0$, for if you see $A_h L$ and $L A_c$ as complex matrices, let $v \in W_c \otimes \mathbb{C}$ be an eigenvector of A_c

So $A_c v = \lambda v$ with necessarily $\lambda \in i\mathbb{R} \subset \mathbb{C}$ (recall $A_c = A|_{W_c}$)

But then $A_h(Lv) = A_h L v = L A_c v$
$= L\lambda v = \lambda(Lv)$.

So either $Lv = 0$ or Lv is an eigenvector of A_h for eigenvalue $\lambda \in i\mathbb{R}$. But $A_h := A|_{W_h}$ only has eigenvalues in $\mathbb{C} - i\mathbb{R}$, so $Lv = 0$ ✱. Hence, L vanishes on all eigenvectors of A_c, and one shows similarly that L vanishes on generalised eigenvectors of A_c. Hence $L = 0$ as predicted in the theory

⋮

k th order $\quad A_h \Psi^{(k)}(x_c) - D\Psi^{(k)}(x_c) A_c x_c \quad (✱✱✱)$

$= $ k th order $\{$ ✱ $- G_c(x_c, \Psi(x_c)) + D\Psi(x_c) G_k(x_c, \Psi(x_c))\}$

these are all terms you solved for in steps $k-1, k-2 \cdots 1$.

It can be shown that (✱✱✱) has a unique solution $\Psi^{(k)}(x_c)$. see

[The equation $(g(x))x\,Ax - Bg(x) = h(x)$, H. K. Wimmer J. math Anal. Appl 67, 1979]

So the Taylor coefficients of Ψ (in so far as they are defined) are unique, even for local centre manifolds.

(14) However, a local centre manifold itself is not unique (if the conditions of global centre manifolds are not fulfilled, i.e. if you define a centre manifold as invariant, going through $(0,0)$ and a graph over W_c to W_h), see the next example:

Ex. 2) Consider the ODE on $\mathbb{R}^2$

[From: Centre manifolds, normal forms and elementary bifurcations by A. Vanderbauwhede]

$$\dot{x} = -x^3$$
$$\dot{y} = -y$$

As before, $W_c = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$
$W_h = \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$

For any $\alpha, \beta \in \mathbb{R}$ (not necessarily the same) consider the

$$\psi_{\alpha,\beta}(x) = \begin{cases} \alpha e^{-\frac{1}{2x^2}} & \text{for } x < 0 \\ 0 & \text{if } x = 0 \\ \beta e^{-\frac{1}{2x^2}} & \text{for } x > 0 \end{cases}$$

Note, if $(\alpha,\beta) \neq (\alpha',\beta') \in \mathbb{R}^2$, then for any open neighborhood U of $(0,0)$ we have $\psi_{\alpha,\beta}|_U \neq \psi_{\alpha',\beta'}|_U$ as functions.

Yet, they are all 'center manifolds' as for $x \neq 0$ and $\gamma = \alpha$ or β (whichever appropriate) (write $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x,y)$)

$$F(x, \psi_{\alpha,\beta}(x)) = \begin{pmatrix} -x^3 \\ -\gamma e^{-\frac{1}{2x^2}} \end{pmatrix}, \quad T_{(x+\psi_{\alpha,\beta}(x))} M_c = \left\{ \begin{pmatrix} v \\ \frac{\gamma e^{-\frac{1}{2x^2}}}{x^3} \cdot v \end{pmatrix} \right\}_{v \in \mathbb{R}}$$

(15) Choosing $V = -x^3$, we see that indeed $F(x, \psi_{\alpha,\beta}(x)) \in T_{(x,\psi_{\alpha,\beta}(x))} M_c$, so that M_c is invariant for any choice of α, β.

Moreover, writing $F(x,y) = \begin{pmatrix} -x^3 \\ -y \end{pmatrix}$, we see that $F(-x,-y) = -F(x,y)$.

The condition that M_c is invariant under this $(\mathbb{Z}/2\mathbb{Z})$-symmetry is (for $x<0$)

$$(x, \alpha e^{-\frac{1}{2x^2}}) \in M_c \Rightarrow (-x, -\alpha e^{-\frac{1}{2x^2}}) \in M_c$$

$$(-x, \beta e^{-\frac{1}{2(-x)^2}}) \in M_c$$

So this symmetry is only respected if $\alpha = -\beta$, so not always (and F has even more symmetries not always respected).

$=$

Back to symmetry, suppose $F(x) = Ax + G(x)$ satisfies the conditions of the theorem, so that $\psi : W_c \to W_H$ exists satisfying

$$M_c := \{ (x_c, \psi(x_c)) \in W_c \oplus W_H \mid x_c \in W_c \} \subset \mathbb{R}^{N=n}$$

$$= \{ x \in \mathbb{R}^n \mid \sup_{t \in \mathbb{R}} \| \pi_H f_t(x) \| < \infty \}$$

with $f_t(x)$ the flow of $\dot{x} = F(x)$ through x (i.e. $f_0(x) = x$). Suppose also that F admits a group symmetry: $F \circ C_g = C_g \circ F$

(16) $\forall\ g \in G$. If we define a center-manifold as an flow invariant graph of a map $\Psi: W_c \to W_h$ satisfying $\Psi(0)=0$, then we've seen that not every center manifold necessarily shares the symmetry, i.e. satisfies $gM_c \subseteq M_c\ \forall g \in G$. However, this can be said for the specific center manifold M_c given to us by the theorem:

Prop: Suppose $F(x) = Ax + G(x)$ (with $A := DF(0)$) satisfies $F \circ B = B \circ F$ for some matrix $B \in \mathbb{R}^{n\times n}$ (not necessarily invertible), then we have:

1) $BW_c (:= \{Bx_c \mid x_c \in W_c\}) \subseteq W_c$ & $BW_h \subseteq W_h$ (with W_c and W_h determined by $A = DF(0)$)

2) $\pi_c \circ B = B \circ \pi_c$, $\pi_h \circ B = B \circ \pi_h$ (with π_c and π_h determined by the splitting $\mathbb{R}^n = W_c \oplus W_h$)

3) If $\sup_{t\in\mathbb{R}} \| \pi_h \varphi_t(x) \| < \infty$ then

$\sup_{t\in\mathbb{R}} \| \pi_h \varphi_t(Bx) \| < \infty$

4) If a map Ψ exist from W_c to W_h such that $\{(x_c, \Psi(x_c)) \mid x_c \in W_c\}$

$= \{ x \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} \| \pi_h \varphi_t(x) \| < \infty \}$

(17) Then $\Psi \circ B = B \circ \Psi$ (Note that this makes sense by point 1))

5) For $R : W_c \circlearrowleft$ defined by $R(x_c) = \pi_c F(x_c + \Psi(x_c))$ we have $R \circ B = B \circ R$ (again, makes sense by point 1), i.e. $BW_c \subseteq W_c$)

Proof 1) [sketch] complexify, and use observations like: if $Av = \lambda v$ then $A(Bv) = ABv = BAv = B\lambda v = \lambda(Bv)$ so Bv is either 0 or an eigenvector for the eigenvalue λ. (Alternatively, play with expressions $\tilde{A}_\lambda := (A - \lambda Id)^n (A - \bar{\lambda} Id)^n$ and note that $\tilde{A}_\lambda \circ B = B \circ \tilde{A}_\lambda$ and so $B(\ker(\tilde{A}_\lambda)) \subseteq \ker(\tilde{A}_\lambda)$

2) Given $x \in \mathbb{R}^n$, write $x = \pi_c(x) + \pi_h(x)$ and so $Bx = B\pi_c(x) + B\pi_h(x)$.
By 1), $B\pi_c(x) \in W_c$, $B\pi_h(x) \in W_h$ so (from $Bx = \pi_c(Bx) + \pi_h(Bx)$, trivially) we get $B\pi_c(x) = \pi_c(Bx)$ and $B\pi_h(x) = \pi_h(Bx)$.
(Note also that $\pi_h = Id_{\mathbb{R}^n} - \pi_c$)

3) Denote by $\|B\|$ the operatornorm of B, suppose $\|\pi_h \varphi_t(x)\| < C$ for some $C \in \mathbb{R}_{>0}$ and $\forall t \in \mathbb{R}$
We claim that $\varphi_t(Bx) = B\varphi_t(x)$, this follows by uniqueness of the flow, as ...

(18) $\frac{d}{dt}(B\varphi_t(x)) = B\,F(\varphi_t(x)) = F(B\varphi_t(x))$

so $B\varphi_t(x) = \varphi_t(y)$ where $y = B\varphi_t(x)\big|_{t=0}$.

so $y = B\varphi_0(x) = Bx$, so $B\varphi_t(x) = \varphi_t(Bx)$

Hence $\|\pi_h \varphi_t(Bx)\| = \|\pi_h B\varphi_t(x)\| = \|B\pi_h\varphi_t(x)\|$ (by 2)

$\leq \|B\|\cdot\|\pi_h\varphi_t(x)\| < \|B\|C < \infty$

4) Write $M_c := \{x \in \mathbb{R}^n \mid \sup_{t\in\mathbb{R}} \|\pi_h\varphi_t(x)\| < \infty\}$

$= \{x_c + \Psi(x_c) \mid x_c \in W_c\}$

then by 3), $x \in M_c \Rightarrow Bx \in M_c$

now pick $x_c \in W_c$, then $x_c + \Psi(x_c) \in M_c$

so $B(x_c + \Psi(x_c)) = Bx_c + B\Psi(x_c) \in M_c$

Also, $x_c \in W_c \Rightarrow Bx_c \in W_c$ (by 1)) so

$(Bx_c) + \Psi(Bx_c) \in M_c$. Summarising:

We have both

$Bx_c + B\Psi(x_c) \in M_c$

$Bx_c + \Psi B(x_c) \in M_c$

with $Bx_c \in W_c$, $B\Psi(x_c), \Psi B(x_c) \in W_h$ (by 1))

as M_c is a graph of $\Psi: W_c \to W_h$, we conclude that $B\Psi(x_c) = \Psi B(x_c)$, as $x_c \in W_c$ was chosen freely, we find $B\circ\Psi = \Psi\circ B$

5) $R(Bx_c) = \pi_c F(Bx_c + \Psi(Bx_c))$

$= \pi_c F(Bx_c + B\Psi(x_c))$ by 4)

$= \pi_c F(B(x_c + \Psi(x_c))$

$= \pi_c B F(x_c + \Psi(x_c))$

by 2) $= B\pi_c F(x_c + \Psi(x_c)) = B R(x_c)$ □

(19) Example (a non-invertible symmetry):

Consider the coupled cell network ODE on $\mathbb{R}^3$ given by

$$\dot{X}_1 = f(X_1, X_2, X_3)$$
$$\dot{X}_2 = f(X_2, X_3, X_3) \quad \text{for some } f: \mathbb{R}^3 \to \mathbb{R}$$
$$\dot{X}_3 = f(X_3, X_3, X_3),$$

corresponding to the classical 3-cell feedforward network:



(We can arrange for it to satisfy conditions needed for the theorem on centre manifold reduction by adapting f (If $f(x) = Df_{(x)} + g(x) \Rightarrow \hat{f}(x) = Df_{(x)} + \chi(\|x\|)g(x)$

Define the linear map $A: \mathbb{R}^3 \to \mathbb{R}^3$

$$A(X_1, X_2, X_3) = (X_2, X_3, X_3) \quad \text{and write}$$

$$F(X) = \begin{pmatrix} f(X_1, X_2, X_3) \\ f(X_2, X_3, X_3) \\ f(X_3, X_3, X_3) \end{pmatrix} \quad \text{where } X = (X_1, X_2, X_3)$$

then

$$F(AX) = \begin{pmatrix} f(X_2, X_3, X_3) \\ f(X_3, X_3, X_3) \\ f(X_3, X_3, X_3) \end{pmatrix} \quad \& \quad AF(X) = \begin{pmatrix} f(X_2, X_3, X_3) \\ f(X_3, X_3, X_3) \\ f(X_3, X_3, X_3) \end{pmatrix}$$

so $F \circ A = A \circ F$ (and note that A is non-invertible)

In fact, it is not hard to see that if $G : \mathbb{R}^3 \to \mathbb{R}^3$ is any vectorfield satisfying $G \circ A = A \circ G$ then a function $g : \mathbb{R}^3 \to \mathbb{R}$ exists such that

$$G(X) = \begin{pmatrix} g(X_1, X_2, X_3) \\ g(X_2, X_3, X_3) \\ g(X_3, X_3, X_3) \end{pmatrix}$$

Back to $F(X)$, we see that

$$DF(0) = \begin{pmatrix} a & b & c \\ 0 & a & b+c \\ 0 & 0 & a+b+c \end{pmatrix}$$

where $a = \frac{\partial f}{\partial x}(0)$, $b = \frac{\partial f}{\partial y}(0)$, $c = \frac{\partial f}{\partial z}(0)$ for $f(x, y, z)$.

The eigenvalues are a (2 times) and $a+b+c$ (1 time), so let's assume $a = 0$.

We get W_c = "generalised kernel"
$= \{(x, y, 0) \mid x, y \in \mathbb{R}\} \leq \mathbb{R}^3$
(with W_h dependent on b & c)

By our result, A maps W_c to itself, and the reduced vectorfield R on W_c also satisfies $R \circ A = A \circ R$.

Indeed, $A(x, y, 0) = (y, 0, 0) \in W_c$, so let us see what R can look like:

(21) A acts on W_c by sending an element $(x, y, 0)$ to $(y, 0, 0)$. Using x, y as coordinates on W_c, we write

$R(x,y) = \begin{pmatrix} h(x,y) \\ \ell(x,y) \end{pmatrix}$, so that

$R(A(x,y)) = \begin{pmatrix} h(y,0) \\ \ell(y,0) \end{pmatrix}$ and $AR(x,y) = \begin{pmatrix} \ell(x,y) \\ 0 \end{pmatrix}$

From $R \circ A = A \circ R$ we therefore get $\begin{matrix} \ell(x,y) = h(y,0) \\ \& \quad 0 = \ell(y,0) \end{matrix}$

so that

$$\boxed{R(x,y) = \begin{pmatrix} h(x,y) \\ h(y,0) \end{pmatrix} \quad \text{with } h(0,0) = 0}$$

Note that many features of the original system $F(x) = \begin{pmatrix} f(x_1, x_2, x_3) \\ f(x_2, x_3, x_3) \\ f(x_3, x_3, x_3) \end{pmatrix}$ are preserved in R

i.e. in R the y coordinate does not depend on the x-coordinate (as in F x_3 does not depend on x_2 and x_1, and x_2 does not depend on x_1) so we still have something "feedforward-like" and we have the same function $h: \mathbb{R}^2 \to \mathbb{R}$ describing the behavior of both x & y.

In fact, re-embedding W_c into $\mathbb{R}^3$ as $(x,y) \mapsto (x,y,0)$ we obtain the vectorfield

$$R(x,y) = \begin{pmatrix} h(x,y) \\ h(y,0) \\ 0 \end{pmatrix} \overset{(h(0,0)=0)}{=\!=} \begin{pmatrix} h(x,y) \\ h(y,0) \\ h(0,0) \end{pmatrix} = \begin{pmatrix} \hat{h}(x,y,0) \\ \hat{h}(y,0,0) \\ \hat{h}(0,0,0) \end{pmatrix}$$

" " "

$R(x,y,0)$

where $\hat{h} : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\hat{h}(x,y,z) := h(x,y) \quad \text{(for instance)}$$

Hence R is of the same form as F!

(So "feed-forward structure" can be preserved" in centre manifold reduction)

So how about our favorite example

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(y) \end{pmatrix}$? (with in general f & g "having nothing to do with each other")

- setting $F(x,y) = \begin{pmatrix} f(x,y) \\ g(y) \end{pmatrix}$ no linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ exists such that (for general f & g)

  $F \circ A = A \circ F$, except $A = Id_{\mathbb{R}^2}$

- However, write $F(x,y) = \begin{pmatrix} f(x,y) \\ g(y) \end{pmatrix}$ and define

  $A : \mathbb{R}^2 \to \mathbb{R}$, $A(x,y) = y$

  then $A \circ F = g \circ A$.

  (as $AF(x,y) = g(y) = g(A(x,y))$)

- Moreover, let's fix $A : \mathbb{R}^2 \to \mathbb{R}$, $A(x,y) = y$ and try to determine all $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R} \to \mathbb{R}$ s.t. $A \circ F = g \circ A$

(23) We get, for $F(x,y) = \begin{pmatrix} f(x,y) \\ h(x,y) \end{pmatrix}$,

$$A \underset{(x,y)}{\cdot} F = h(x,y), \quad g \cdot A(x,y) = g(y)$$

Hence, $A \cdot F = g \cdot A \iff h(x,y) = g(y)$

$$\iff F(x,y) = \begin{pmatrix} f(x,y) \\ g(y) \end{pmatrix}$$

So, we need to generalise our definition of symmetry to encorporate these situations. The right generalisation is a quiver symmetry.

First a recap of classical symmetry, we have there

① A group, which is an algebraic structure

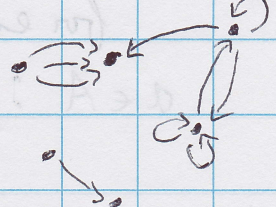
② (Linear) maps for each element of the group (satisfying certain consistency conditions)

③ A dynamical system (say) for which the symmetry maps respect the dynamics

Generalising, we get:

Def

① A quiver is a directed graph, (so a set of Nodes N, a set of arrows A, and source and target maps $A \underset{t}{\overset{s}{\rightrightarrows}} N$)



Usually N & A are finite sets, but no other restrictions apply (i.e. (multiple) self-loops are allowed, multiple arrows with the same source and target - disconnected components, etc.)

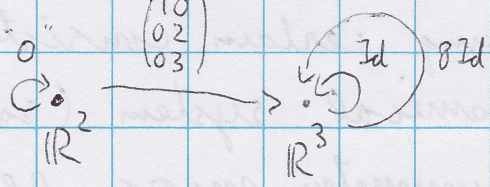
(24) (2) A quiver representation of a quiver $Q = (A \overset{s}{\underset{t}{\rightrightarrows}} N)$ is a choice of

(fix a field, here always $\mathbb{R}$) → vectorspace V_p for each $p \in N$ and a choice of linear map $B_a : V_{s(a)} \to V_{t(a)}$ for each arrow a, from $V_{s(a)}$ to $V_{t(a)}$ with $s(a) \in N$ source of a, $t(a) \in N$ target of a.

$$\underset{V_p}{p} \xrightarrow[B_a]{a} \underset{V_q}{q}$$

Other than the consistency condition that the domain & codomain make sense.

(so not necessarily injective or surjective or bijective, no condition on $B_a \circ B_{\hat{a}}$ for $a, \hat{a} \in A$ if this makes sense, etc.)



(3) Given a quiver Q together with a representation of it $((V_p)_{p \in N}, (B_a)_{a \in A})$, a quiver equivariant (or symmetric) system is a choice of vectorfields F_p on V_p for each node $p \in N$, s.t for each arrow $a \in A$:

$$F_{t(a)} \circ B_a = B_a \circ F_{s(a)}$$

i.e.

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{B_a} & V_{t(a)} \\ \downarrow{\scriptstyle F_{s(a)}} & \circlearrowright & \downarrow{\scriptstyle F_{t(a)}} \\ V_{s(a)} & \xrightarrow{B_a} & V_{t(a)} \end{array}$$

commutes

(25) **Example 1** Let Q be the quiver with 2 vertices and 1 arrow, arranged as

$$Q = \quad \textcircled{1} \xrightarrow{a} \textcircled{2}$$

We consider the representation

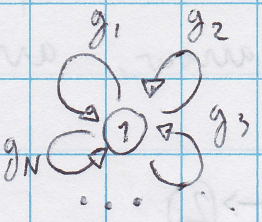
$$\textcircled{1} \xrightarrow{a} \textcircled{2}$$
$$\mathbb{R}^2 \qquad B_a(x,y) = y \qquad \mathbb{R}$$

We have already seen that the symmetric vectorfields for Q with this representation are precisely given by

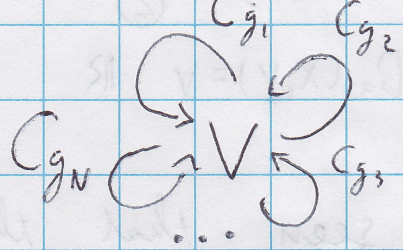
$$\textcircled{1} \xrightarrow{a} \textcircled{2}$$
$$\mathbb{R}^2 \qquad B_a(x,y) = y \qquad \mathbb{R}$$
$$\begin{pmatrix} f(x,y) \\ g(y) \end{pmatrix} \qquad (g(y))$$

for any $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ (so $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $F_1(x,y) = \begin{pmatrix} f(x,y) \\ g(y) \end{pmatrix}$, $F_2 : \mathbb{R} \to \mathbb{R}$, $F_2(z) = g(z)$, then $B_a \circ F_1 = F_2 \circ B_a$)

Example 2 Let G be a finite group, $\#G = N$, and $(V, (\ell_g : V \to V)_{g \in G})$ a representation of G on a finite-dimensional vector space V. Consider the quiver Q_G with 1 node and N arrow, indexed by elements of G ...

$Q_G =$ 

We get the representation of Q_G induced by $(V, (C_g)_{g \in G})$:



Symmetric vectorfields for Q_G with this representation are precisely vectorfields

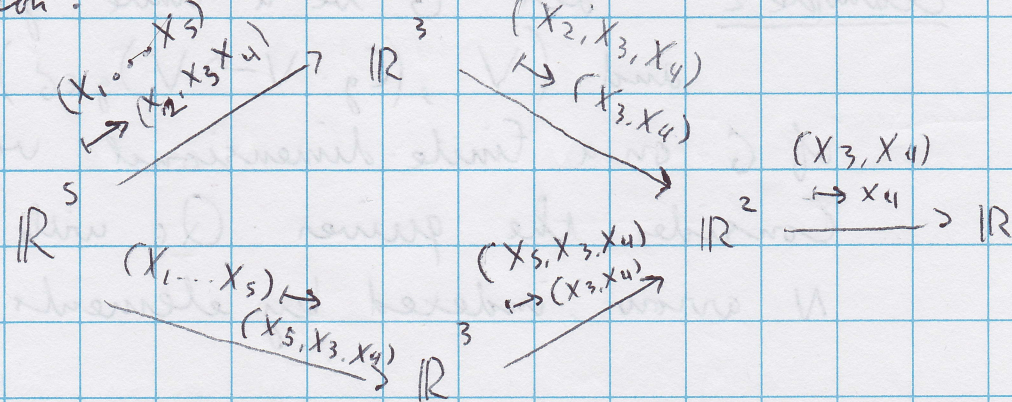
$F : V \to V$ s.t. $F \circ C_g = C_g \circ F \quad \forall g \in G$.

, so G-equivariant vectorfields in the classical case

Note that the quiver Q_G has a lot of representations not induced by representations of G, as for the quiver we do not require

$C_e = Id$ (e unit of G) nor $C_g \circ C_h = C_{(gh)}$, $g, h \in G$.

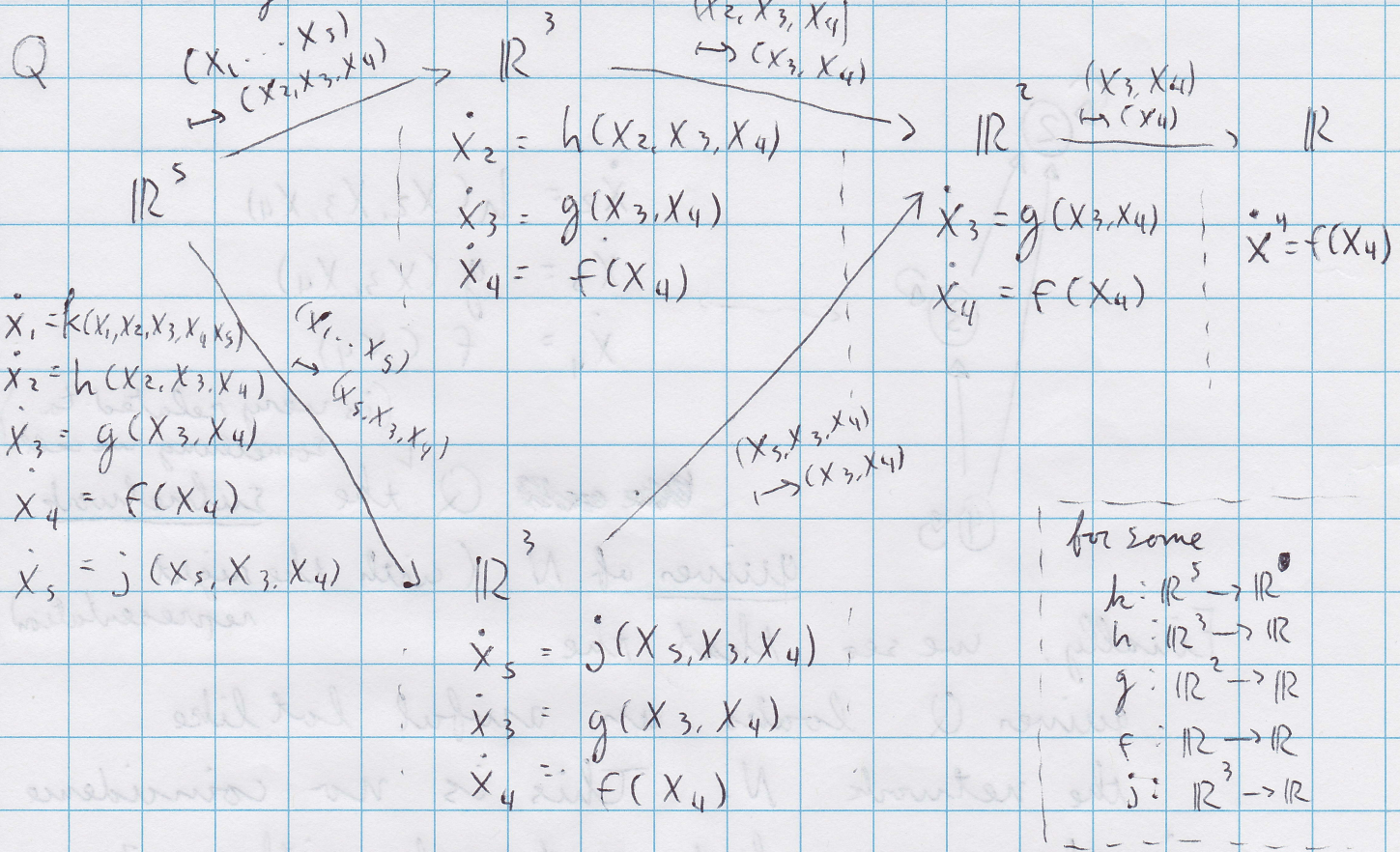
Example 3. Consider the quiver with representation:



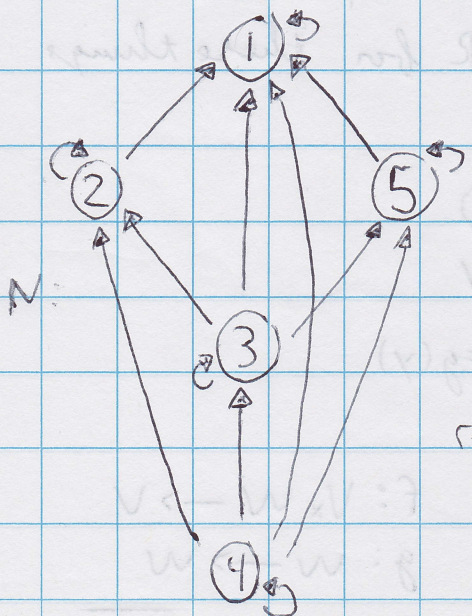
where the labelling of some variables is

intensionally chosen to be a bit weird.
We may find all equivariant vectorfields solving "from right to left".

we get



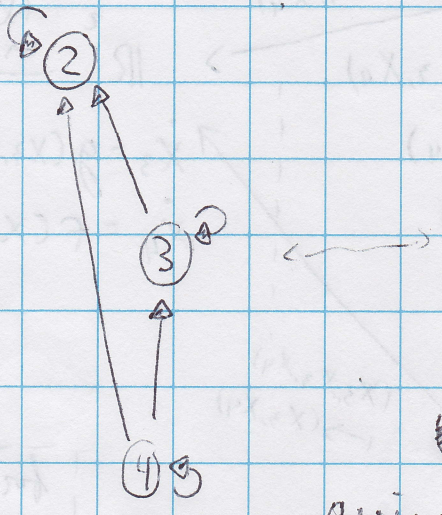
We are of course mostly interested in the ODE at the first node from the left, it gives precisely all <u>admissible vectorfields for the network</u>:



$$\dot{X}_1 = k(X_1, X_2, X_3, X_4, X_5)$$
$$\dot{X}_2 = h(X_2, X_3, X_4)$$
$$\dot{X}_3 = g(X_3, X_4)$$
$$\dot{X}_4 = f(X_4)$$
$$\dot{X}_5 = j(X_5, X_3, X_4)$$

(with all different nodes and all different arrows)

Note that the other vertices in the quiver correspond to subnetworks of N, e.g.



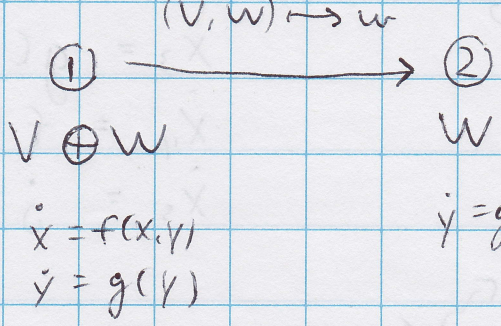
$$\dot{x}_2 = h(x_2, x_3, x_4)$$
$$\dot{x}_3 = g(x_3, x_4)$$
$$\dot{x}_4 = f(x_4)$$

(is very related to something we call)

~~We call~~ Q the <u>subnetwork quiver</u> of N (with the right representation)

Finally, we see that the quiver Q looks an awful lot like the network N. This is no coincidence in this case, but not a rule either. In general, it is best not to confuse Q and N (as objects they are very different)

<u>Example 4</u> The internal phase space of each node need not be $\mathbb{R}$ for these things to work:



for vector spaces (general) V, W and $f: V \times W \to V$, $g: W \to W$

(29) Our main theorem can be extended to quivers in the following straightforward way.

Thm [Center manifold reduction for systems with quiver symmetry, from "Quiver representations and dimension reduction in dynamical systems", Bob Rink, Soren Schwenker, & Eric Nyhalt]

Let $Q=(N, A, s, t)$ be a quiver, $R=((V_i)_{i\in N}, (B_a)_{a\in A})$ a representation, and $F=(F_i)_{i\in N}$ a symmetric system for Q and R. then

$DF = (D_0F_i)_{i\in N}$ (the Jacobians at $0\in V_i, \forall i\in N$) forms a linear quiver symmetric system for Q and R, Moreover:

1) If $W_c^i, W_h^i \subseteq V_i$ are the centre- and hyperbolic subspaces of D_0F_i, for $i\in N$, then for all arrows $a\in A$, (with source node $s(a)\in N$, and target node $t(a)\in N$) we have

$$B_a(W_c^{s(a)}) \subseteq W_c^{t(a)} \quad \& \quad B_a(W_h^{s(a)}) \subseteq W_h^{t(a)}$$

2) If $\pi_c^i : V_i = W_c^i \oplus W_h^i \to W_c^i$ and $\pi_h^i : V_i \to W_h^i$ are the corresponding projections then

$$B_a\pi_c^{s(a)} = \pi_c^{t(a)}B_a \quad \& \quad B_a\pi_h^{s(a)} = \pi_c^{t(a)}B_a, \ \forall a\in A.$$

In other words, $\pi_c := (\pi_c^i)_{i\in N}$ & $\pi_h := (\pi_h^i)_{i\in N}$ are linear quiver sym. maps for Q, R.

3) Let $\varphi_t^i(x)$ denote the flow of F_i on V_i, then: (assuming flows are defined for all t).

$$\sup_{t\in\mathbb{R}} \| \pi_h^{s(a)} \varphi_t^{s(a)}(x)\| < \infty \implies \sup_{t\in\mathbb{R}} \| \pi_h^{t(a)} \varphi_t^{t(a)}(B_a x)\| < \infty$$

4) If $\Psi_i : W_c^i \to W_h^i$ exists for which

$$\mathrm{Graph}(\Psi_i) := \{ x_c + \Psi_i(x_c) \mid x_c \in W_c^i \} = \{ x \in V_i \mid \sup_{t \in \mathbb{R}} \| \pi_h^i \varphi_t(x) \| < \infty \}$$

(for all $i \in N$) Then

$$B_a \circ \Psi_{s(a)} = \Psi_{t(a)} \circ B_a \quad \text{for all arrows } a \in A$$

5) The Reduced vectorfields R_i on W_c^i, defined by

$R_i(x_c) = \pi_c^i F_i(x_c + \Psi_i(x_c))$ satisfy

$$B_a \circ R_{s(a)} = R_{t(a)} \circ B_a \quad \text{for all arrows } a \in A$$

Proof [Sketch] If $B_a \circ F_{s(a)} = F_{t(a)} \circ B_a$, then differentiation at $0 \in V_{s(a)}$ gives:

$$B_a \circ D_0 F_{s(a)} = D_0 F_{t(a)} \circ B_a$$

1) Precisely as in the proof of linear maps between the same space (See page 16-17)
Note for instance that if $(D_0 F_{s(a)}) v = \lambda v$ then $(D_0 F_{t(a)})(B_a v) = B_a (D_0 F_{s(a)}) v = B_a \circ \lambda . v = \lambda (B_a v)$
$(\lambda \in \mathbb{C},\ v \in V \otimes \mathbb{C})$

2) Let $x \in V_{s(a)}$, write $x = x_c + x_h$ with $x_c \in W_c^{s(a)}$, $x_h \in W_h^{s(a)}$, then $B_a x = \underbrace{B_a x_c}_{\in W_c^{t(a)}} + \underbrace{B_a x_h}_{\in W_h^{t(a)}}$
[by point 1)]

So $\pi_c^{t(a)}(B_a x) = B_a x_c = B_a(\pi_c^{s(a)} x)$, and so forth.

3) Use that $B_a \varphi_t^{s(a)}(x) = \varphi_t^{t(a)}(B_a x)$, so

$$\begin{aligned} & \| \pi_h^{t(a)} \varphi_t^{t(a)} (B_a x) \| \\ = & \| \pi_h^{t(a)} B_a \varphi_t^{s(a)}(x) \| \\ = & \| B_a \pi_h^{s(a)} \varphi_t^{s(a)}(x) \| \\ \leq & \| B_a \| \cdot \| \pi_h^{s(a)} \varphi_t^{s(a)}(x) \| \end{aligned}$$

4) $x_c + \Psi_{s(a)}(x_c) \in M_c^{s(a)} \Rightarrow \underbrace{B_a x_c}_{\in W_c^{t(a)}} + \underbrace{B_a \Psi_{s(a)}(x_c)}_{\in W_h^{t(a)}} \in M_c^{t(a)}$