The Picard-Lindelöf Theorem: Existence and Uniqueness of Solutions

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We treat an important result on the local existence and uniqueness of solutions to ODEs, called the Picard-Lindelöf theorem. The exposition here strongly follows that of [1].

We first need the concept of a function that is locally Lipschitz in x:

Definition 1. *Let U be an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ a function. We write a point in U as (t, x) with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. The function F is called* locally Lipschitz in x *if for every $(t_0, x_0) \in U$ there exist an open set $V \subseteq U$ with $(t_0, x_0) \in V$ and a number $C > 0$ such that*

$$\|F(t,x) - F(s,y)\| \leq C\|x - y\| \,, \tag{1}$$

for all $(t, x), (s, y) \in V$.

We then have:

Theorem 1 (The Picard-Lindelöf Theorem)**.** *Let U be an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ a continuous function that is locally Lipschitz in x. Given $(t_0, x_0) \in U$, there exists an $\epsilon > 0$ and a continuously differentiable function $\gamma : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^n$ such that for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$ we have $(t, \gamma(t)) \in U$ and*

$$\frac{d}{dt}\gamma(t) = F(t, \gamma(t)) \,. \tag{2}$$

Moreover, let $I_1, I_2 \subseteq \mathbb{R}$ be two open intervals and $\gamma_i : I_i \to \mathbb{R}^n$ for $i \in \{1, 2\}$ two continuously differentiable functions such that $(t, \gamma_i(t)) \in U$ and

$$\frac{d}{dt}\gamma_i(t) = F(t, \gamma_i(t)) \,, \tag{3}$$

for all $t \in I_i$, for $i \in \{1, 2\}$. If $\gamma_1(s) = \gamma_2(s)$ for some $s \in I_1 \cap I_2$ then $\gamma_1(t) = \gamma_2(t)$ for all $t \in I_1 \cap I_2$.

The proof of Theorem 1 uses a so-called contraction argument. To this end, we need:

Definition 2. *Let (X, d) be a metric space. A function $G : X \to X$ is called a* contraction *if there exists a positive real number $\mu < 1$ such that*

$$d(G(x), G(y)) \leq \mu d(x, y)\,, \tag{4}$$

for all $x, y \in X$.

Lemma 1. *Let (X, d) be a complete metric space and $G : X \to X$ a contraction. There is a unique $y \in X$ such that $G(y) = y$. Moreover, for every $x \in X$ the limit $\lim_{n\to\infty} G^n(x)$ exists and is given by y.*

Proof. We first show uniqueness of the fixed point of G, assuming one exists. Suppose $y_1, y_2 \in X$ satisfy $G(y_1) = y_1$ and $G(y_2) = y_2$. Then

$$d(y_1, y_2) = d(G(y_1), G(y_2)) \leq \mu d(y_1, y_2)\,. \tag{5}$$

Thus

$$(1 - \mu)d(y_1, y_2) \leq 0 \tag{6}$$

and, since $\mu < 1$, we obtain

$$d(y_1, y_2) \leq 0\,. \tag{7}$$

Of course $d(y_1, y_2) \geq 0$ and so $d(y_1, y_2) = 0$, from which we see that $y_1 = y_2$.

We now fix $x \in X$ and consider the sequence

$$x, G(x), G^2(x), \dots\,. \tag{8}$$

If $G(x) = x$ then all elements of this sequence are the same and so the limit exists and is given by x. Suppose therefore that $G(x) \neq x$, so that $d(x, G(x)) \neq 0$, and let $\epsilon > 0$ be given. Since $\mu < 1$, there exists an $N \in \mathbb{N}$ such that

$$\mu^N < \frac{\epsilon(1 - \mu)}{d(x, G(x))}\,. \tag{9}$$

Now, given any $m, n \geq N$ with $m > n$, we have

$$\begin{aligned} d(G^n(x), G^m(x)) &\leq d(G^n(x), G^{n+1}(x)) + \cdots + d(G^{m-1}(x), G^m(x)) \\ &\leq \mu^n d(x, G(x)) + \cdots + \mu^{m-1} d(x, G(x)) \\ &= \mu^n d(x, G(x))(1 + \mu + \mu^2 + \cdots + \mu^{m-1-n}) \\ &\leq \mu^n d(x, G(x))(1 + \mu + \mu^2 + \dots) \\ &= \mu^n d(x, G(x)) \frac{1}{1 - \mu} \leq \mu^N d(x, G(x)) \frac{1}{1 - \mu} < \epsilon\,. \end{aligned} \tag{10}$$

Thus the sequence in (8) is Cauchy and, since X is complete, it has a limit $z = z(x) \in X$. Next, we fix $\epsilon' > 0$ and let $N > 0$ be such that $n \geq N$ implies $d(G^n(x), z) < \frac{\epsilon'}{1+\mu}$. It follows that

$$\begin{aligned} d(z, G(z)) &\leq d(z, G^{N+1}(x)) + d(G^{N+1}(x), G(z)) \\ &\leq d(z, G^{N+1}(x)) + \mu d(G^N(x), z) < \frac{\epsilon'}{1 + \mu}(1 + \mu) = \epsilon'\,. \end{aligned} \tag{11}$$

Thus for any $\epsilon' > 0$ we have $d(z, G(z)) < \epsilon'$. This of course means $d(z, G(z)) = 0$ and so $G(z) = z$. We conclude that at least one element $y \in X$ exists such that $G(y) = y$. (We of course assume X is non-empty, so take any $x \in X$ and let $y = \lim_{n\to\infty} G^n(x)$.) By our first result, such a y is unique. Thus, we find $y = \lim_{n\to\infty} G^n(x)$ for all $x \in X$, which completes the proof. □

To use Lemma 1, we next give an example of a complete metric space (see Lemma 2 below), followed by a contraction (see lemmas 3 and 4 below).

Lemma 2. *Let $I \subseteq \mathbb{R}$ be an open interval containing a point t_0 and $K \subseteq \mathbb{R}^n$ a compact subset containing a point x_0. Define the set*

$$\mathcal{U}_{I,t_0}^{K,x_0} := \{\gamma \colon I \to K \mid \gamma \text{ is continuous and } \gamma(t_0) = x_0\}\,, \tag{12}$$

together with the map

$$d(\gamma_1, \gamma_2) := \sup_{t\in I} \|\gamma_1(t) - \gamma_2(t)\|\,. \tag{13}$$

Then d defines a metric on $\mathcal{U}_{I,t_0}^{K,x_0}$ and $(\mathcal{U}_{I,t_0}^{K,x_0}, d)$ is complete.

Proof. We first show that d defines a metric on $\mathcal{U}_{I,t_0}^{K,x_0}$. Since K is compact, there is a $C > 0$ such that $\|x\| < C$ for all $x \in K$. Therefore $\|\gamma_1(t) - \gamma_2(t)\| < 2C$ for all $t \in I$ and $\gamma_1, \gamma_2 \in \mathcal{U}_{I,t_0}^{K,x_0}$, which in turn shows that $d(\gamma_1, \gamma_2) \in \mathbb{R}_{\geq 0}$.

Next, it is clear from the definition that $d(\gamma_1, \gamma_2) = d(\gamma_2, \gamma_1)$ for all $\gamma_1, \gamma_2 \in \mathcal{U}_{I,t_0}^{K,x_0}$, and that $d(\gamma_1, \gamma_2) = 0$ if and only if $\gamma_1 = \gamma_2$.

Finally, for all $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{U}_{I,t_0}^{K,x_0}$ we have

$$\begin{aligned} d(\gamma_1, \gamma_3) &= \sup_{t\in I} \|\gamma_1(t) - \gamma_3(t)\| = \sup_{t\in I} \|\gamma_1(t) - \gamma_2(t) + \gamma_2(t) - \gamma_3(t)\| \\ &\leq \sup_{t\in I} \Big(\|\gamma_1(t) - \gamma_2(t)\| + \|\gamma_2(t) - \gamma_3(t)\| \Big) \\ &\leq \sup_{t\in I} \|\gamma_1(t) - \gamma_2(t)\| + \sup_{t\in I} \|\gamma_2(t) - \gamma_3(t)\| \\ &= d(\gamma_1, \gamma_2) + d(\gamma_2, \gamma_3)\,. \end{aligned} \tag{14}$$

This shows that d is indeed a well-defined metric on $\mathcal{U}_{I,t_0}^{K,x_0}$.

It remains to show that $(\mathcal{U}_{I,t_0}^{K,x_0}, d)$ is complete. To this end, let $(\gamma_n)_n$ be a Cauchy-sequence of elements in $\mathcal{U}_{I,t_0}^{K,x_0}$. Then for every $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$d(\gamma_n, \gamma_m) := \sup_{t\in I} \|\gamma_n(t) - \gamma_m(t)\| < \epsilon\,, \tag{15}$$

whenever $m, n \geq N_\epsilon$. In particular, for any fixed $s \in I$ we have $\|\gamma_n(s) - \gamma_m(s)\| < \epsilon$ whenever $m, n \geq N_\epsilon$, which shows that $(\gamma_n(s))_n$ is a Cauchy-sequence in K. Since, this latter set is closed, we conclude that $(\gamma_n(s))_n$ has a limit in K, which we will denote by $\gamma(s)$. The limit of $(\gamma_n)_n$ will of course be the function $\gamma : t \mapsto \gamma(t)$, though we need to show that this function is continuous and that the γ_n converge to it.

For the latter statement, fix $\epsilon > 0$ and $s \in I$. Since $\lim_{n\to\infty} \gamma_n(s) = \gamma(s)$, there exists an $M_{s,\epsilon} \in \mathbb{N}$ such that $n > M_{s,\epsilon}$ implies $\|\gamma_n(s) - \gamma(s)\| < 1/3\epsilon$. Let $k > N_{1/3\epsilon}$ and $\ell > \max(N_{1/3\epsilon}, M_{s,\epsilon})$ be given. We find

$$\begin{aligned} \|\gamma_k(s) - \gamma(s)\| &= \|\gamma_k(s) - \gamma_\ell(s) + \gamma_\ell(s) - \gamma(s)\| \\ &\leq \|\gamma_k(s) - \gamma_\ell(s)\| + \|\gamma_\ell(s) - \gamma(s)\| \\ &< 1/3\epsilon + 1/3\epsilon = 2/3\epsilon\,. \end{aligned} \tag{16}$$

We therefore have

$$\sup_{s\in I} \|\gamma_k(s) - \gamma(s)\| < \epsilon \tag{17}$$

whenever $k > N_{1/3\epsilon}$. This shows that $(\gamma_n)_n$ converges to γ, provided we can show that this latter function lies in $\mathcal{U}^{K,x_0}_{I,t_0}$. To this end, note that by definition of γ we have

$$\gamma(t_0) = \lim_{n\to\infty} \gamma_n(t_0) = \lim_{n\to\infty} x_0 = x_0\,.$$

To show that γ is continuous, let $s \in I$ and $\epsilon > 0$ be given and fix any $k \in \mathbb{N}$ for which $\sup_{u\in I} \|\gamma_k(u) - \gamma(u)\| < 1/3\epsilon$. Since γ_k is continuous, there exists a $\delta > 0$ such that $\|\gamma_k(t) - \gamma_k(s)\| < 1/3\epsilon$ for all $t \in I$ with $|t - s| < \delta$. For any such t we have

$$\begin{aligned} \|\gamma(t) - \gamma(s)\| &= \|\gamma(t) - \gamma_k(t) + \gamma_k(t) - \gamma_k(s) + \gamma_k(s) - \gamma(s)\| \\ &\leq \|\gamma(t) - \gamma_k(t)\| + \|\gamma_k(t) - \gamma_k(s)\| + \|\gamma_k(s) - \gamma(s)\| \\ &< 1/3\epsilon + 1/3\epsilon + 1/3\epsilon = \epsilon\,. \end{aligned} \tag{18}$$

Thus γ is indeed continuous and we find $\gamma \in \mathcal{U}^{K,x_0}_{I,t_0}$ as the limit of $(\gamma_n)_n$. This completes the proof. $\square$

Note that $\mathcal{U}^{K,x_0}_{I,t_0}$ is non-empty, as it contains the function that is constantly equal to x_0.

We now assume U is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $F\colon U \to \mathbb{R}^n$ a continuous function that is locally Lipschitz in x, as in the setting of Theorem 1. Given $(t_0, x_0) \in U$, we know that there exists an open set $V \subseteq U$ containing (t_0, x_0) such that

$$\|F(t, x) - F(s, y)\| \leq C\|x - y\|\,, \tag{19}$$

for some $C > 0$ and all $(t, x), (s, y) \in V$. We may now pick a compact subset of V containing (t_0, x_0), which is more specifically of the form $[t_0 - \delta, t_0 + \delta] \times \overline{B(x_0, \delta)}$ for some $\delta > 0$. Here $\overline{B(x_0, \delta)}$ denotes the closed ball in $\mathbb{R}^n$ around x_0:

$$\overline{B(x_0, \delta)} := \{v \in \mathbb{R}^n \mid \|v - x_0\| \leq \delta\}\,. \tag{20}$$

Note that Equation (19) then holds for all $s, t \in [t_0 - \delta, t_0 + \delta]$ and $x, y \in \overline{B(x_0, \delta)}$. As $[t_0 - \delta, t_0 + \delta] \times \overline{B(x_0, \delta)}$ is compact and F continuous, a constant $D > 0$ exists such that

$$\|F(t, x)\| < D \tag{21}$$

for all $t \in [t_0 - \delta, t_0 + \delta]$ and $x \in \overline{B(x_0, \delta)}$. In this setting, we have the result of Lemma 3 below. To make the notation somewhat lighter, we will denote by I_ϵ the open interval $(t_0 - \epsilon, t_0 + \epsilon) \subseteq [t_0 - \delta, t_0 + \delta]$, for any $0 < \epsilon \leq \delta$.

Lemma 3. *Let $0 < \epsilon < \delta, \frac{\delta}{D}$ be given. There is a well-defined operator*

$$\mathcal{L} : \mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0} \to \mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$$

given by

$$\big(\mathcal{L}(\gamma)\big)(t) = x_0 + \int_{t_0}^{t} F(\tau, \gamma(\tau)) d\tau \tag{22}$$

for all $\gamma \in \mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$.

Proof. As both γ and F are assumed continuous with the latter bounded, the integral in Equation (22) is well-defined. It is clear that $\mathcal{L}(\gamma) \colon I_\epsilon \to \mathbb{R}^n$ is a continuous function and that $\big(\mathcal{L}(\gamma)\big)(t_0) = x_0$. It remains to show that $\big(\mathcal{L}(\gamma)\big)(t) \in \overline{B(x_0, \delta)}$ for all $t \in I_\epsilon$. To this end, we note that

$$\begin{aligned} \|\big(\mathcal{L}(\gamma)\big)(t) - x_0\| &= \left\| \int_{t_0}^{t} F(\tau, \gamma(\tau)) d\tau \right\| \leq \left| \int_{t_0}^{t} \|F(\tau, \gamma(\tau))\| d\tau \right| \\ &\leq \left| \int_{t_0}^{t} D d\tau \right| = D|t - t_0| < D\epsilon < \delta \,. \end{aligned} \tag{23}$$

Thus $\mathcal{L}$ indeed maps $\mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$ into itself, which completes the proof. ☐

Definition 3. *We call the operator $\mathcal{L}$ of Lemma 3 the* Picard operator*.*

The result we have been working towards is of course:

Lemma 4. *Let $0 < \epsilon < \delta, \frac{\delta}{D}, \frac{1}{2C}$ be given. Then the Picard operator is a contraction on $\mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$.*

Proof. Given $\gamma_1, \gamma_2 \in \mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$, a direct calculation shows that

$$\begin{aligned} \|\big(\mathcal{L}(\gamma_1)\big)(t) - \big(\mathcal{L}(\gamma_2)\big)(t)\| &= \left\| \int_{t_0}^{t} F(\tau, \gamma_1(\tau)) - F(\tau, \gamma_2(\tau)) d\tau \right\| \\ &\leq \left| \int_{t_0}^{t} \|F(\tau, \gamma_1(\tau)) - F(\tau, \gamma_2(\tau))\| d\tau \right| \\ &\leq \left| \int_{t_0}^{t} C\|\gamma_1(\tau) - \gamma_2(\tau)\| d\tau \right| \\ &\leq |t - t_0| C \sup_{\tau \in I_\epsilon} \|\gamma_1(\tau) - \gamma_2(\tau)\| \\ &< \epsilon C d(\gamma_1, \gamma_2) < \frac{1}{2} d(\gamma_1, \gamma_2) \,. \end{aligned} \tag{24}$$

Taking the supremum over t, we indeed arrive at

$$d(\mathcal{L}(\gamma_1), \mathcal{L}(\gamma_2)) \leq \frac{1}{2} d(\gamma_1, \gamma_2)\,, \tag{25}$$

which completes the proof. □

As a corollary, we obtain

Corollary 1. *Let $F : U \to \mathbb{R}^n$ be a continuous function that is locally Lipschitz in x. Let $(t_0, x_0) \in U$ be given and $\delta > 0$ be such that $(t_0, x_0) \in [t_0 - \delta, t_0 + \delta] \times \overline{B(x_0, \delta)} \subseteq V \subseteq U$ as above. Then for $\epsilon > 0$ small enough, there is precisely one $\gamma \in \mathcal{U}^{\overline{B(x_0,\delta)},x_0}_{I_\epsilon,t_0}$ that is differentiable and satisfies*

$$\frac{d}{dt}\gamma(t) = F(t, \gamma(t)) \tag{26}$$

for all $t \in I_\epsilon$.

Proof. Suppose $\gamma \in \mathcal{U}^{\overline{B(x_0,\delta)},x_0}_{I_\epsilon,t_0}$ is differentiable and satisfies Equation (26). Integrating this identity from t_0 to $t \in I_\epsilon$ gives

$$\gamma(t) - \gamma(t_0) = \gamma(t) - x_0 = \int_{t_0}^{t} F(\tau, \gamma(\tau)) d\tau\,, \tag{27}$$

and so

$$\gamma(t) = x_0 + \int_{t_0}^{t} F(\tau, \gamma(\tau)) d\tau = \big(\mathcal{L}(\gamma)\big)(t)\,. \tag{28}$$

Conversely, if $\gamma = \mathcal{L}(\gamma)$ then

$$\gamma(t) = x_0 + \int_{t_0}^{t} F(\tau, \gamma(\tau)) d\tau\,. \tag{29}$$

This implies that γ is differentiable, by the fundamental theorem of calculus. Moreover, the derivative satisfies Equation (26). In conclusion, $\gamma \in \mathcal{U}^{\overline{B(x_0,\delta)},x_0}_{I_\epsilon,t_0}$ is differentiable and satisfies Equation (26) if and only if it is a fixed point of $\mathcal{L}$. By Lemma 4, $\mathcal{L}$ is a contraction on $\mathcal{U}^{\overline{B(x_0,\delta)},x_0}_{I_\epsilon,t_0}$ for small enough ϵ, so that Lemma 1 indeed gives us a unique fixed point. This completes the proof. □

We now have everything in place to prove the Picard-Lindelöf theorem, Theorem 1.

Proof of Theorem 1. Existence of a local solution γ follows directly from Corollary 1. Since we have

$$\frac{d}{dt}\gamma(t) = F(t, \gamma(t)) \tag{30}$$

for all $t \in I_\epsilon$, it follows that γ is continuously differentiable.

Now suppose we have two solutions $\gamma_i \colon I_i \to \mathbb{R}^n$, $i \in \{1, 2\}$, and suppose $\gamma_1(s) = \gamma_2(s)$ for some $s \in I_1 \cap I_2$. We define the set

$$J := \{t \in I_1 \cap I_2 \mid \gamma_1(t) = \gamma_2(t)\}\,. \tag{31}$$

It is clear that J is a closed subset of the open interval $I_1 \cap I_2$ and, since it contains s, we see that J is non-empty. Now let $t_0 \in J$ be given and write $x_0 = \gamma_1(t_0) = \gamma_2(t_0)$. By Corollary 1, there exist constants $\delta > 0$ and ϵ_1 such that for all $0 < \epsilon < \epsilon_1$, the set $\mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$ contains precisely one function γ that is differientable and solves Equation (30) for $t \in I_\epsilon$. Here as before, we write $I_\epsilon := (t_0 - \epsilon, t_0 + \epsilon)$. Since $\gamma_1(t_0) = \gamma_2(t_0) = x_0$, we may choose $\epsilon_2 > 0$ small enough such that $I_{\epsilon_2} \subseteq I_1 \cap I_2$ and $\gamma_i(I_{\epsilon_2}) \subseteq \overline{B(x_0, \delta)}$ for $i \in \{1, 2\}$. Then if we choose any $0 < \epsilon < \epsilon_1, \epsilon_2$, we see that $\gamma_1|_{I_\epsilon}, \gamma_2|_{I_\epsilon} \in \mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$ both solve Equation (30) for $t \in I_\epsilon$, whereas $\mathcal{U}_{I_\epsilon, t_0}^{\overline{B(x_0,\delta)}, x_0}$ contains only one such solution. We conclude that $\gamma_1|_{I_\epsilon} = \gamma_2|_{I_\epsilon}$ and so $I_\epsilon = (t_0 - \epsilon, t_0 + \epsilon) \subseteq J$. This shows that J is an open set. Thus $J \subseteq I_1 \cap I_2$ is non-empty, open and closed, and so we see that $J = I_1 \cap I_2$. This completes the proof. $\square$

References

[1] Marcelo Viana and José M Espinar. *Differential equations: a dynamical systems approach to theory and practice*, volume 212. American Mathematical Society, 2021.